

GENERALIZED DESCRIPTIVE SET THEORY AND CLASSIFICATION THEORY

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ABSTRACT. Descriptive set theory is mainly concerned with studying subsets of the space of all countable binary sequences. In this paper we study the generalization where countable is replaced by uncountable. We explore properties of generalized Baire and Cantor spaces, equivalence relations and their Borel reducibility. The study shows that the descriptive set theory looks very different in this generalized setting compared to the classical, countable case. We also draw the connection between the stability theoretic complexity of first-order theories and the descriptive set theoretic complexity of their isomorphism relations. Our results suggest that Borel reducibility on uncountable structures is a model theoretically natural way to compare the complexity of isomorphism relations.

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1. HISTORY AND MOTIVATION

There is a long tradition in studying connections between Borel structure of Polish spaces (descriptive set theory) and model theory. The connection arises from the fact that any class of countable structures can be coded into a subset of the space 2^ω provided all structures in the class have domain ω . A survey on this topic is given in [8]. Suppose X and Y are subsets of 2^ω and let E_1 and E_2 be equivalence relations on X and Y respectively. If $f: X \rightarrow Y$ is a map such that $E_1(x, y) \iff E_2(f(x), f(y))$, we say that f is a *reduction of E_1 to E_2* . If there

exists a Borel or continuous reduction, we say that E_1 is Borel or continuously *reducible* to E_2 , denoted $E_1 \leq_B E_2$ or $E_1 \leq_c E_2$. The mathematical meaning of this is that f *classifies E_1 -equivalence in terms of E_2 -equivalence*.

The benefit of various reducibility and irreducibility theorems is roughly the following. A reducibility result, say $E_1 \leq_B E_2$, tells us that E_1 is at most as complicated as E_2 ; once you understand E_2 , you understand E_1 (modulo the reduction). An irreducibility result, $E_1 \not\leq_B E_2$ tells that there is no hope in trying to classify E_1 in terms of E_2 , at least in a “Borel way”. From the model theoretic point of view, the isomorphism relation, and the elementary equivalence relation (in some language) on some class of structures are the equivalence relations of main interest. But model theory in general does not restrict itself to countable structures. Most of stability theory and Shelah’s classification theory characterizes first-order theories in terms of their uncountable models. This leads to the generalization adopted in this paper. We consider the space 2^κ for an uncountable cardinal κ with the idea that models of size κ are coded into elements of that space.

This approach, to connect such uncountable descriptive set theory with model theory, began in the early 1990’s. One of the pioneering papers was by Mekler and Väänänen [22]. A survey on the research done in 1990’s can be found in [34] and a discussion of the motivational background for this work in [33]. A more recent account is given the book [35], Chapter 9.6.

Let us explain how our approach differs from the earlier ones and why it is useful. For a first-order complete countable theory in a countable vocabulary T and a cardinal $\kappa \geq \omega$, define

$$S_T^\kappa = \{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models T\} \text{ and } \cong_T^\kappa = \{(\eta, \xi) \in (S_T^\kappa)^2 \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$$

where $\eta \mapsto \mathcal{A}_\eta$ is some fixed coding of (all) structures of size κ . We can now define the partial order on the set of all theories as above by

$$T \leq^\kappa T' \iff \cong_T^\kappa \leq_B \cong_{T'}^\kappa .$$

As pointed out above, $T \leq^\kappa T'$ says that \cong_T^κ is at most as difficult to classify as $\cong_{T'}^\kappa$. But does this tell us whether T is a simpler theory than T' ? Rough answer: *If $\kappa = \omega$, then no but if $\kappa > \omega$, then yes.*

To illustrate this, let $T = \text{Th}(\mathbb{Q}, \leq)$ be the theory of the order of the rational numbers (DLO) and let T' be the theory of a vector space over the field of

rational numbers. Without loss of generality we may assume that they are models of the same vocabulary.

It is easy to argue that the model class defined by T' is strictly simpler than that of T . (For instance there are many questions about T , unlike T' , that cannot be answered in ZFC; say existence of a saturated model.) On the other hand $\cong_T^\omega \leq_B \cong_{T'}^\omega$ and $\cong_{T'}^\omega \not\leq_B \cong_T^\omega$ because there is only one countable model of T and there are infinitely many countable models of T' . But for $\kappa > \omega$ we have $\cong_T^\kappa \not\leq_B \cong_{T'}^\kappa$ and $\cong_{T'}^\kappa \leq_B \cong_T^\kappa$, since there are 2^κ equivalence classes of \cong_T^κ and only one equivalence class of $\cong_{T'}^\kappa$. Another example, introduced in Martin Koerwien's Ph.D. thesis and his article [18] shows that there exists an ω -stable theory without DOP and without OTOP with depth 2 for which \cong_T^ω is not Borel, while we show here that for $\kappa > \omega$, \cong_T^κ is Borel for all classifiable shallow theories.

The results suggest that the order \leq^κ for $\kappa > \omega$ corresponds naturally to the classification of theories in stability theory: the more complex a theory is from the viewpoint of stability theory, the higher it seems to sit in the ordering \leq^κ and vice versa. Since dealing with uncountable cardinals often implies the need for various cardinality or set theoretic assumptions beyond ZFC, the results are not always as simple as in the case $\kappa = \omega$, but they tell us a lot. For example, our results easily imply the following (modulo some mild cardinality assumptions on κ):

- If T is deep and T' is shallow, then $\cong_T \not\leq_B \cong_{T'}$.
- If T is unstable and T' is classifiable, then $\cong_T \not\leq_B \cong_{T'}$.

2. INTRODUCTION

2.1. Notations and Conventions.

2.1.1. *Set Theory.* We use standard set theoretical notation:

- $A \subset B$ means that A is a subset of B or is equal to B .
- $A \subsetneq B$ means proper subset.
- Union, intersection and set theoretical difference are denoted respectively by $A \cup B$, $A \cap B$ and $A \setminus B$. For larger unions and intersections $\bigcup_{i \in I} A_i$ etc.
- $\mathcal{P}(A)$ is the power set of A and $[A]^{<\kappa}$ is the set of subsets of A of size $< \kappa$

Usually the Greek letters κ, λ and μ will stand for cardinals and α, β and γ for ordinals, but this is not strict. Also η, ξ, ν are usually elements of κ^κ or 2^κ and p, q, r are elements of $\kappa^{<\kappa}$ or $2^{<\kappa}$. $\text{cf}(\alpha)$ is the cofinality of α (the least ordinal β for which there exists an increasing unbounded function $f: \beta \rightarrow \alpha$).

By S_λ^κ we mean $\{\alpha < \kappa \mid \text{cf}(\alpha) = \lambda\}$. A λ -*cub set* is a subset of a limit ordinal (usually of cofinality $> \lambda$) which is unbounded and contains suprema of all bounded increasing sequences of length λ . A set is *cub* if it is λ -cub for all λ . A set is *stationary* if it intersects all cub sets and λ -*stationary* if it intersects all λ -cub sets. Note that $C \subset \kappa$ is λ -cub if and only if $C \cap S_\lambda^\kappa$ is λ -cub and $S \subset \kappa$ is λ -stationary if and only if $S \cap S_\lambda^\kappa$ is (just) stationary.

If (\mathbb{P}, \leq) is a forcing notion, we write $p \leq q$ if p and q are in \mathbb{P} and q forces more than p . Usually \mathbb{P} is a set of functions equipped with inclusion and $p \leq q \iff p \subset q$. In that case \emptyset is the weakest condition and we write $\mathbb{P} \Vdash \varphi$ to mean $\emptyset \Vdash_{\mathbb{P}} \varphi$.

2.1.2. Functions. We denote by $f(x)$ the value of x under the mapping f and by $f[A]$ or just fA the image of the set A under f . Similarly $f^{-1}[A]$ or just $f^{-1}A$ indicates the inverse image of A . Domain and range are denoted respectively by $\text{dom } f$ and $\text{ran } f$.

If it is clear from the context that f has an inverse, then f^{-1} denotes that inverse. For a map $f: X \rightarrow Y$ *injective* means the same as *one-to-one* and *surjective* the same as *onto*.

Suppose $f: X \rightarrow Y^\alpha$ is a function with range consisting of sequences of elements of Y of length α . The projection pr_β is a function $Y^\alpha \rightarrow Y$ defined by $\text{pr}_\beta((y_i)_{i<\alpha}) = y_\beta$. For the coordinate functions of f we use the notation $f_\beta = \text{pr}_\beta \circ f$ for all $\beta < \alpha$.

By support of a function f we mean the subset of $\text{dom } f$ in which f takes non-zero values, whatever “zero” means depending on the context (hopefully never unclear). The support of f is denoted by $\text{sprt } f$.

2.1.3. Model Theory. In Section 2.2.3 we fix a countable vocabulary and assume that all theories are theories in this vocabulary. Moreover we assume that they are first-order, complete and countable. By $\text{tp}(\bar{a}/A)$ we denote the complete type of $\bar{a} = (a_1, \dots, a_{\text{length } \bar{a}})$ over A where $\text{length } \bar{a}$ is the length of the sequence \bar{a} .

We think of models as tuples $\mathcal{A} = \langle \text{dom } \mathcal{A}, P_n^{\mathcal{A}} \rangle_{n<\omega}$ where the P_n are relation symbols in the vocabulary and the $P_n^{\mathcal{A}}$ are their interpretations. If a relation R has arity n (a property of the vocabulary), then for its interpretation it holds that $R^{\mathcal{A}} \subset (\text{dom } \mathcal{A})^n$. In Section 2.2.3 we adopt more conventions concerning this.

In Section 4.2.1 and Chapter 5 we will use the following stability theoretical notions stable, superstable, DOP, OTOP, shallow and $\kappa(T)$. Classifiable means

superstable with no DOP nor OTOP, the least cardinal in which T is stable is denoted by $\lambda(T)$.

2.1.4. Reductions. Let $E_1 \subset X^2$ and $E_2 \subset Y^2$ be equivalence relations on X and Y respectively. A function $f: X \rightarrow Y$ is a *reduction* of E_1 to E_2 if for all $x, y \in X$ we have that $x E_1 y \iff f(x) E_2 f(y)$. Suppose in addition that X and Y are topological spaces. Then we say that E_1 is *continuously reducible* to E_2 , if there exists a continuous reduction from E_1 to E_2 and we say that E_1 is *Borel reducible* to E_2 if there is a Borel reduction. For the definition of Borel adopted in this paper, see Definition 15. We denote the fact that E_1 is continuously reducible to E_2 by $E_1 \leq_c E_2$ and respectively Borel reducibility by $E_1 \leq_B E_2$.

We say that relations E_2 and E_1 are (Borel) *bireducible* to each other if $E_2 \leq_B E_1$ and $E_1 \leq_B E_2$.

2.2. Ground Work.

2.2.1. Trees and Topologies. Throughout the paper κ is assumed to be an uncountable regular cardinal which satisfies

$$\kappa^{<\kappa} = \kappa \quad (*)$$

(For justification of this, see below.) We look at the space κ^κ , i.e. the functions from κ to κ and the space formed by the initial segments $\kappa^{<\kappa}$. It is useful to think of $\kappa^{<\kappa}$ as a tree ordered by inclusion and of κ^κ as a topological space of the branches of $\kappa^{<\kappa}$; the topology is defined below. Occasionally we work in 2^κ and $2^{<\kappa}$ instead of κ^κ and $\kappa^{<\kappa}$.

1. Definition. A *tree* t is a partial order with a root in which the sets $\{x \in t \mid x < y\}$ are well ordered for each $y \in t$. A *branch* in a tree is a maximal linear suborder.

A tree is called a $\kappa\lambda$ -*tree*, if there are no branches of length λ or higher and no element has $\geq \kappa$ immediate successors. If t and t' are trees, we write $t \leq t'$ to mean that there exists an order preserving map $f: t \rightarrow t'$, $a <_t b \Rightarrow f(a) <_{t'} f(b)$.

Convention. Unless otherwise said, by a tree $t \subset (\kappa^{<\kappa})^n$ we mean a tree with domain being a downward closed subset of

$$(\kappa^{<\kappa})^n \cap \{(p_0, \dots, p_{n-1}) \mid \text{dom } p_0 = \dots = \text{dom } p_{n-1}\}$$

ordered as follows: $(p_0, \dots, p_{n-1}) < (q_0, \dots, q_{n-1})$ if $p_i \subset q_i$ for all $i \in \{0, \dots, n-1\}$. It is always a $\kappa^+, \kappa + 1$ -tree.

2. **Example.** Let $\alpha < \kappa^+$ be an ordinal and let t_α be the tree of descending sequences in α ordered by end extension. The root is the empty sequence. It is a $\kappa^+\omega$ -tree. Such t_α can be embedded into $\kappa^{<\omega}$, but note that not all subtrees of $\kappa^{<\omega}$ are $\kappa^+\omega$ -trees (there are also $\kappa^+, \omega + 1$ -trees).

In fact the trees $\kappa^{<\beta}$, $\beta \leq \kappa$ and t_α are universal in the following sense:

Fact ($\kappa^{<\kappa} = \kappa$). *Assume that t is a $\kappa^+, \beta + 1$ -tree, $\beta \leq \kappa$ and t' is $\kappa^+\omega$ -tree. Then*

- (1) *there is an embedding $f: t \rightarrow \kappa^{<\beta}$,*
- (2) *and a strictly order preserving map $f: t' \rightarrow t_\alpha$ for some $\alpha < \kappa^+$ (in fact there is also such an embedding f).* \square

Define the topology on κ^κ as follows. For each $p \in \kappa^{<\kappa}$ define the basic open set

$$N_p = \{\eta \in \kappa^\kappa \mid \eta \upharpoonright \text{dom}(p) = p\}.$$

Open sets are precisely the empty set and the sets of the form $\bigcup X$, where X is a collection of basic open sets. Similarly for 2^κ .

There are many justifications for the assumption $(*)$ which will be most apparent after seeing the proofs of our theorems. The crucial points can be summarized as follows: if $(*)$ does not hold, then

- the space κ^κ does not have a dense subset of size κ ,
- there are open subsets of κ^κ that are not κ -unions of basic open sets which makes controlling Borel sets difficult (see Definition 15 on page 14).
- Vaught's generalization of the Lopez-Escobar theorem (Theorem 24) fails, see Remark 25 on page 21.
- The model theoretic machinery we are using often needs this cardinality assumption (see e.g. Theorem 30 and proof of Theorem 72).

Initially the motivation to assume $(*)$ was simplicity. Many statements concerning the space $\kappa^{<\kappa}$ are independent of ZFC and using $(*)$ we wanted to make the scope of such statements neater. In the statements of (important) theorems we mention the assumption explicitly.

Because the intersection of less than κ basic open sets is either empty or a basic open set, we get the following.

Fact ($\kappa^{<\kappa} = \kappa$). *The following hold for a topological space $P \in \{2^\kappa, \kappa^\kappa\}$:*

- (1) The intersection of less than κ basic open sets is either empty or a basic open set,
- (2) The intersection of less than κ open sets is open,
- (3) Basic open sets are closed,
- (4) $|\{A \subset P \mid A \text{ is basic open}\}| = \kappa$,
- (5) $|\{A \subset P \mid A \text{ is open}\}| = 2^\kappa$.

In the space $\kappa^\kappa \times \kappa^\kappa = (\kappa^\kappa)^2$ we define the ordinary product topology.

3. Definition. A set $Z \subset \kappa^\kappa$ is Σ_1^1 if it is a projection of a closed set $C \subset (\kappa^\kappa)^2$. A set is Π_1^1 if it is the complement of a Σ_1^1 set. A set is Δ_1^1 if it is both Σ_1^1 and Π_1^1 .

As in standard descriptive set theory ($\kappa = \omega$), we have the following:

4. Theorem. For $n < \omega$ the spaces $(\kappa^\kappa)^n$ and κ^κ are homeomorphic. \square

Remark. This standard theorem can be found for example in Jech's book [15]. Applying this theorem we can extend the concepts of Definition 3 to subsets of $(\kappa^\kappa)^n$. For instance a subset A of $(\kappa^\kappa)^n$ is Σ_1^1 if for a homeomorphism $h: (\kappa^\kappa)^n \rightarrow \kappa^\kappa$, $h[A]$ is Σ_1^1 according to Definition 3.

2.2.2. Ehrenfeucht-Fraïssé Games. We will need Ehrenfeucht-Fraïssé games in various connections. It serves also as a way of coding isomorphisms.

5. Definition (Ehrenfeucht-Fraïssé games). Let t be a tree, κ a cardinal and \mathcal{A} and \mathcal{B} structures with domains A and B respectively. Note that t might be an ordinal. The game $\text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ is played by players **I** and **II** as follows. Player **I** chooses subsets of $A \cup B$ and climbs up the tree t and player **II** chooses partial functions $A \rightarrow B$ as follows. Suppose a sequence

$$(X_i, p_i, f_i)_{i < \gamma}$$

has been played (if $\gamma = 0$, then the sequence is empty). Player **I** picks a set $X_\gamma \subset A \cup B$ of cardinality strictly less than κ such that $X_\delta \subset X_\gamma$ for all ordinals $\delta < \gamma$. Then player **I** picks a $p_\gamma \in t$ which is $<_t$ -above all p_δ where $\delta < \gamma$. Then player **II** chooses a partial function $f_\gamma: A \rightarrow B$ such that $X_\gamma \cap A \subset \text{dom } f_\gamma$, $X_\gamma \cap B \subset \text{ran } f_\gamma$, $|\text{dom } f_\gamma| < \kappa$ and $f_\delta \subset f_\gamma$ for all ordinals $\delta < \gamma$. The game ends when player **I** cannot go up the tree anymore, i.e. $(p_i)_{i < \gamma}$ is a branch. Player **II** wins if

$$f = \bigcup_{i < \gamma} f_i$$

is a partial isomorphism. Otherwise player **I** wins.

A *strategy* of player **II** in $\text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ is a function

$$\sigma: ([A \cup B]^{<\kappa} \times t)^{<\text{ht}(t)} \rightarrow \bigcup_{I \in [A]^{<\kappa}} B^I,$$

where $[R]^{<\kappa}$ is the set of subsets of R of size $< \kappa$ and $\text{ht}(t)$ is the *height* of the tree, i.e.

$$\text{ht}(t) = \sup\{\alpha \mid \alpha \text{ is an ordinal and there is an order preserving embedding } \alpha \rightarrow t\}.$$

A strategy of **I** is similarly a function

$$\tau: \left(\bigcup_{I \in [A]^{<\kappa}} B^I \right)^{<\text{ht}(t)} \rightarrow [A \cup B]^{<\kappa} \times t.$$

We say that a strategy τ of player **I** *beats* strategy σ of player **II** if the play $\tau * \sigma$ is a win for **I**. The play $\tau * \sigma$ is just the play where **I** uses τ and **II** uses σ . Similarly σ beats τ if $\tau * \sigma$ is a win for **II**. We say that a strategy is a *winning strategy* if it beats all opponents strategies.

The notation $X \uparrow \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ means that player X has a winning strategy in $\text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$.

Remark. By our convention $\text{dom } \mathcal{A} = \text{dom } \mathcal{B} = \kappa$, so while player **I** picks a subset of $\text{dom } \mathcal{A} \cup \text{dom } \mathcal{B}$ he actually just picks a subset of κ , but as a small analysis shows, this does not alter the game.

Consider the game $\text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$, where $|\mathcal{A}| = |\mathcal{B}| = \kappa$, $|t| \leq \kappa$ and $\text{ht}(t) \leq \kappa$. The set of strategies can be identified with κ^κ , for example as follows. The moves of player **I** are members of $[A \cup B]^{<\kappa} \times t$ and the moves of player **II** are members of $\bigcup_{I \in [A]^{<\kappa}} B^I$. By our convention $\text{dom } \mathcal{A} = \text{dom } \mathcal{B} = A = B = \kappa$, so these become $V = [\kappa]^{<\kappa} \times t$ and $U = \bigcup_{I \in [\kappa]^{<\kappa}} \kappa^I$. By our cardinality assumption $\kappa^{<\kappa} = \kappa$, these sets are of cardinality κ .

Let

$$\begin{aligned} f: U &\rightarrow \kappa \\ g: U^{<\kappa} &\rightarrow \kappa \\ h: V &\rightarrow \kappa \\ k: V^{<\kappa} &\rightarrow \kappa \end{aligned}$$

be bijections. Let us assume that $\tau: U^{<\kappa} \rightarrow V$ is a strategy of player **I** (there cannot be more than κ moves in the game because we assumed $\text{ht}(t) \leq \kappa$). Let

$\nu_\tau: \kappa \rightarrow \kappa$ be defined by

$$\nu_\tau = h \circ \tau \circ g^{-1}$$

and if $\sigma: V^{<\kappa} \rightarrow U$ is a strategy of player **II**, let ν_σ be defined by

$$\nu_\sigma = f \circ \sigma \circ k^{-1}.$$

We say that ν_τ codes τ .

6. Theorem ($\kappa^{<\kappa} = \kappa$). *Let $\lambda \leq \kappa$ be a cardinal. The set*

$$C = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \text{ codes a w.s. of } \mathbf{II} \text{ in } \text{EF}_\lambda^\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)\} \subset (\kappa^\kappa)^3$$

*is closed. If $\lambda < \kappa$, then also the corresponding set for player **I***

$$D = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \text{ codes a w.s. of } \mathbf{I} \text{ in } \text{EF}_\lambda^\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)\} \subset (\kappa^\kappa)^3$$

is closed.

Remark. Compare to Theorem 13.

Proof. Assuming $(\nu_0, \eta_0, \xi_0) \notin C$, we will show that there is an open neighbourhood U of (ν_0, η_0, ξ_0) such that $U \subset (\kappa^\kappa)^3 \setminus C$. Denote the strategy that ν_0 codes by σ_0 . By the assumption there is a strategy τ of **I** which beats σ_0 . Consider the game in which **I** uses τ and **II** uses σ_0 .

Denote the γ^{th} move in this game by (X_γ, h_γ) where $X_\gamma \subset A_{\eta_0} \cup A_{\xi_0}$ and $h_\gamma: A_{\eta_0} \rightarrow A_{\xi_0}$ are the moves of the players. Since player **I** wins this game, there is $\alpha < \lambda$ for which h_α is not a partial isomorphism between \mathcal{A}_{η_0} and \mathcal{A}_{ξ_0} . Let

$$\varepsilon = \sup(X_\alpha \cup \text{dom } h_\alpha \cup \text{ran } h_\alpha)$$

(Recall $\text{dom } \mathcal{A}_\eta = A_\eta = \kappa$ for any η by convention.) Let π be the coding function defined in Definition 12 on page 12. Let

$$\beta_1 = \pi[\varepsilon^{<\omega}] + 1.$$

The idea is that $\eta_0 \restriction \beta_1$ and $\xi_0 \restriction \beta_1$ decide the models \mathcal{A}_{η_0} and \mathcal{A}_{ξ_0} as far as the game has been played. Clearly $\beta_1 < \kappa$.

Up to this point, player **II** has applied her strategy σ_0 precisely to the sequences of the moves made by her opponent, namely to $S = \{(X_\gamma)_{\gamma < \beta} \mid \beta < \alpha\} \subset \text{dom } \sigma_0$. We can translate this set to represent a subset of the domain of ν_0 : $S' = k[S]$, where k is as defined before the statement of the present theorem. Let $\beta_2 = (\sup S') + 1$ and let

$$\beta = \max\{\beta_1, \beta_2\}.$$

Thus $\eta_0 \restriction \beta$, $\xi_0 \restriction \beta$ and $\nu_0 \restriction \beta$ decide the moves $(h_\gamma)_{\gamma < \alpha}$ and the winner.

Now

$$\begin{aligned} U &= \{(\nu, \eta, \xi) \mid \nu \restriction \beta = \nu_0 \restriction \beta \wedge \eta \restriction \beta = \eta_0 \restriction \beta \wedge \xi \restriction \beta = \xi_0 \restriction \beta\} \\ &= N_{\nu_0 \restriction \beta} \times N_{\eta_0 \restriction \beta} \times N_{\xi_0 \restriction \beta}. \end{aligned}$$

is the desired neighbourhood. Indeed, if $(\nu, \eta, \xi) \in U$ and ν codes a strategy σ , then τ beats σ on the structures $\mathcal{A}_\eta, \mathcal{A}_\xi$, since the first α moves are exactly as in the corresponding game of the triple (ν_0, η_0, ξ_0) .

Let us now turn to D . The proof is similar. Assume that $(\nu_0, \eta_0, \xi_0) \notin D$ and ν_0 codes strategy τ_0 of player **I**. Then there is a strategy of **II**, which beats τ_0 . Let $\beta < \kappa$ be, as before, an ordinal such that all moves have occurred before β and the relations of the substructures generated by the moves are decided by $\eta_0 \restriction \beta, \xi_0 \restriction \beta$ as well as the strategy τ_0 . Unlike for player **I**, the win of **II** is determined always only in the end of the game, so β can be $\geq \lambda$. This is why we made the assumption $\lambda < \kappa$, by which we can always have $\beta < \kappa$ and so

$$\begin{aligned} U &= \{(\nu, \eta, \xi) \mid \nu \restriction \beta = \nu_0 \restriction \beta \wedge \eta \restriction \beta = \eta_0 \restriction \beta \wedge \xi \restriction \beta = \xi_0 \restriction \beta\} \\ &= N_{\nu_0 \restriction \beta} \times N_{\eta_0 \restriction \beta} \times N_{\xi_0 \restriction \beta}. \end{aligned}$$

is an open neighbourhood of (ν_0, η_0, ξ_0) in the complement of D . \square

Let us list some theorems concerning Ehrenfeucht-Fraïssé games which we will use in the proofs.

7. Definition. Let T be a theory and \mathcal{A} a model of T of size κ . The $L_{\infty\kappa}$ -Scott height of \mathcal{A} is

$$\sup\{\alpha \mid \exists \mathcal{B} \models T(\mathcal{A} \not\cong \mathcal{B} \wedge \mathbf{II} \upharpoonright \text{EF}_{t_\alpha}^\kappa(\mathcal{A}, \mathcal{B}))\},$$

if the supremum exists and ∞ otherwise, where t_α is as in Example 2 and the subsequent Fact.

Remark. Sometimes the Scott height is defined in terms of quantifier ranks, but this gives an equivalent definition by Theorem 9 below.

8. Definition. The *quantifier rank* $R(\varphi)$ of a formula $\varphi \in L_{\infty\infty}$ is an ordinal defined by induction on the length of φ as follows. If φ quantifier free, then $R(\varphi) = 0$. If $\varphi = \exists \bar{x} \psi(\bar{x})$, then $R(\varphi) = R(\psi(\bar{x})) + 1$. If $\varphi = \neg \psi$, then $R(\varphi) = R(\psi)$. If $\varphi = \bigwedge_{\alpha < \lambda} \psi_\alpha$, then $R(\varphi) = \sup\{R(\psi_\alpha) \mid \alpha < \lambda\}$.

9. Theorem. *Models \mathcal{A} and \mathcal{B} satisfy the same $L_{\infty\kappa}$ -sentences of quantifier rank $< \alpha$ if and only if $\mathbf{II} \uparrow \text{EF}_{t_\alpha}^\kappa(\mathcal{A}, \mathcal{B})$.* \square

The following theorem is a well known generalization of a theorem of Karp [16]:

10. Theorem. *Models \mathcal{A} and \mathcal{B} are $L_{\infty\kappa}$ -equivalent if and only if $\mathbf{II} \uparrow \text{EF}_\omega^\kappa(\mathcal{A}, \mathcal{B})$.* \square

11. Remark. Models \mathcal{A} and \mathcal{B} of size κ are $L_{\kappa+\kappa}$ -equivalent if and only if they are $L_{\infty\kappa}$ -equivalent. For an extensive and detailed survey on this and related topics, see [35].

2.2.3. Coding Models. There are various degrees of generality to which the content of this text is applicable. Many of the results generalize to vocabularies with infinitary relations or to uncountable vocabularies, but not all. We find it reasonable though to fix the used vocabulary to make the presentation clearer.

Models can be coded to models with just one binary predicate. Function symbols often make situations unnecessarily complicated from the point of view of this paper.

Thus our approach is, without great loss of generality, to fix our attention to models with finitary relation symbols of all finite arities.

Let us fix L to be the countable relational vocabulary consisting of the relations P_n , $n < \omega$, $L = \{P_n \mid n < \omega\}$, where each P_n is an n -ary relation: the interpretation of P_n is a set consisting of n -tuples. We can assume without loss of generality that the domain of each L -structure of size κ is κ , i.e. $\text{dom } \mathcal{A} = \kappa$. If we restrict our attention to these models, then the set of all L -models has the same cardinality as κ^κ .

We will next present the way we code the structures and the isomorphisms between them into the elements of κ^κ (or equivalently – as will be seen – to 2^κ).

12. Definition. Let π be a bijection $\pi: \kappa^{<\omega} \rightarrow \kappa$. If $\eta \in \kappa^\kappa$, define the structure \mathcal{A}_η to have $\text{dom}(\mathcal{A}_\eta) = \kappa$ and if $(a_1, \dots, a_n) \in \text{dom}(\mathcal{A}_\eta)^n$, then

$$(a_1, \dots, a_n) \in P_n^{\mathcal{A}_\eta} \iff \eta(\pi(a_1, \dots, a_n)) > 0.$$

In that way the rule $\eta \mapsto \mathcal{A}_\eta$ defines a surjective (onto) function from κ^κ to the set of all L -structures with domain κ . We say that η codes \mathcal{A}_η .

Remark. Define the equivalence relation on κ^κ by $\eta \sim \xi \iff \text{sprt } \eta = \text{sprt } \xi$, where sprt means support, see Section 2.1.2 on page 5. Now we have $\eta \sim \xi \iff \mathcal{A}_\eta = \mathcal{A}_\xi$, i.e. the identity map $\kappa \rightarrow \kappa$ is an isomorphism between \mathcal{A}_η and \mathcal{A}_ξ when $\eta \sim \xi$ and vice versa. On the other hand $\kappa^\kappa / \sim \cong 2^\kappa$, so the coding can be seen also as a bijection between models and the space 2^κ .

The distinction will make little difference, but it is convenient to work with both spaces depending on context. To illustrate the insignificance of the choice between κ^κ and 2^κ , note that \sim is a closed equivalence relation and identity on 2^κ is bireducible with \sim on κ^κ (see Definition 2.1.4).

2.2.4. Coding Partial Isomorphisms. Let $\xi, \eta \in \kappa^\kappa$ and let p be a bijection $\kappa \rightarrow \kappa \times \kappa$. Let $\nu \in \kappa^\alpha$, $\alpha \leq \kappa$. The idea is that for $\beta < \alpha$, $p_1(\nu(\beta))$ is the image of β under a partial isomorphism and $p_2(\nu(\beta))$ is the inverse image of β . That is, for a $\nu \in \kappa^\alpha$, define a relation $F_\nu \subset \kappa \times \kappa$:

$$(\beta, \gamma) \in F_\nu \iff (\beta < \alpha \wedge p_1(\nu(\beta)) = \gamma) \vee (\gamma < \alpha \wedge p_2(\nu(\gamma)) = \beta).$$

If ν happens to be such that F_ν is a partial isomorphism $\mathcal{A}_\xi \rightarrow \mathcal{A}_\eta$, then we say that ν codes a partial isomorphism between \mathcal{A}_ξ and \mathcal{A}_η , this isomorphism being determined by F_ν . If $\alpha = \kappa$ and ν codes a partial isomorphism, then F_ν is an isomorphism and we say that ν codes an isomorphism.

13. Theorem. *The set*

$$C = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \text{ codes an isomorphism between } \mathcal{A}_\eta \text{ and } \mathcal{A}_\xi\}$$

is a closed set.

Proof. Suppose that $(\nu, \eta, \xi) \notin C$ i.e. ν does not code an isomorphism $\mathcal{A}_\eta \cong \mathcal{A}_\xi$. Then (at least) one of the following holds:

- (1) F_ν is not a function,
- (2) F_ν is not one-to-one,
- (3) F_ν does not preserve relations of $\mathcal{A}_\eta, \mathcal{A}_\xi$.

(Note that F_ν is always onto if it is a function and $\text{dom } \nu = \kappa$.) If (1), (2) or (3) holds for ν , then respectively (1), (2) or (3) holds for any triple (ν', η', ξ') where $\nu' \in N_{\nu \upharpoonright \gamma}$, $\eta' \in N_{\eta \upharpoonright \gamma}$ and $\xi' \in N_{\xi \upharpoonright \gamma}$, so it is sufficient to check that (1), (2) or (3) holds for $\nu \upharpoonright \gamma$ for some $\gamma < \kappa$, because

Let us check the above in the case that (3) holds. The other cases are left to the reader. Suppose (3) holds. There is $(a_0, \dots, a_{n-1}) \in (\text{dom } \mathcal{A}_\eta)^n = \kappa^n$ such that

$(a_0, \dots, a_{n-1}) \in P_n$ and $(a_0, \dots, a_{n-1}) \in P_n^{\mathcal{A}_\eta}$ and $(F_\nu(a_0), \dots, F_\nu(a_{n-1})) \notin P_n^{\mathcal{A}_\xi}$.
Let β be greater than

$$\max(\{\pi(a_0, \dots, a_{n-1}), \pi(F_\nu(a_0), \dots, F_\nu(a_{n-1}))\} \\ \cup \{a_0, \dots, a_{n-1}, F_\nu(a_0), \dots, F_\nu(a_{n-1})\}).$$

Then it is easy to verify that any $(\eta', \xi', \nu') \in N_{\eta \upharpoonright \beta} \times N_{\xi \upharpoonright \beta} \times N_{\nu \upharpoonright \beta}$ satisfies (3) as well. \square

14. Corollary. *The set $\{(\eta, \xi) \in (\kappa^\kappa)^2 \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$ is Σ_1^1 .*

Proof. It is the projection of the set C of Theorem 13. \square

2.3. Generalized Borel Sets.

15. Definition. We have already discussed Δ_1^1 sets which generalize Borel subsets of Polish space in one way. Let us see how else can we generalize usual Borel sets to our setting.

- [4, 22] The collection of λ -Borel subsets of κ^κ is the smallest set, which contains the basic open sets of κ^κ and is closed under complementation and under taking intersections of size λ . Since we consider only κ -Borel sets, we write $\text{Borel} = \kappa\text{-Borel}$.
- The collection $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$.
- [4, 22] The collection of *Borel** subsets of κ^κ . A set A is *Borel** if there exists a $\kappa^+ \kappa$ -tree t in which each increasing sequence of limit order type has a unique supremum and a function

$$h: \{\text{branches of } t\} \rightarrow \{\text{basic open sets of } \kappa^\kappa\}$$

such that $\eta \in A \iff$ player **II** has a winning strategy in the game $G(t, h, \eta)$. The game $G(t, h, \eta)$ is defined as follows. At the first round player **I** picks a minimal element of the tree, on successive rounds he picks an immediate successor of the last move played by player **II** and if there is no last move, he chooses an immediate successor of the supremum of all previous moves. Player **II** always picks an immediate successor of the Player **I**'s choice. The game ends when the players cannot go up the tree anymore, i.e. have chosen a branch b . Player **II** wins, if $\eta \in h(b)$. Otherwise **I** wins.

A *dual* of a *Borel** set B is the set

$$B^d = \{\xi \mid \mathbf{I} \upharpoonright G(t, h, \xi)\}$$

where t and h satisfy the equation $B = \{\xi \mid \mathbf{II} \uparrow G(t, h, \xi)\}$. The dual is not unique.

Remark. Suppose that t is a $\kappa^+\kappa$ tree and $h: \{\text{branches of } t\} \rightarrow \text{Borel}^*$ is a labeling function taking values in Borel^* sets instead of basic open sets. Then $\{\eta \mid \mathbf{II} \uparrow G(t, h, \eta)\}$ is a Borel^* set.

Thus if we change the basic open sets to Borel^* sets in the definition of Borel^* , we get Borel^* .

16. *Remark.* Blackwell [2] defined Borel^* sets in the case $\kappa = \omega$ and showed that in fact $\text{Borel} = \text{Borel}^*$. When κ is uncountable it is not the case. But it is easily seen that if t is a $\kappa^+\omega$ -tree, then the Borel^* set coded by t (with some labeling h) is a Borel set, and vice versa: each Borel set is a Borel^* set coded by a $\kappa^+\omega$ -tree. We will use this characterization of Borel.

It was first explicitly proved in [22] that these are indeed generalizations:

17. **Theorem** ([22], $\kappa^{<\kappa} = \kappa$). $\text{Borel} \subset \Delta_1^1 \subset \text{Borel}^* \subset \Sigma_1^1$,

Proof. (Sketch) If A is Borel^* , then it is Σ_1^1 , intuitively, because $\eta \in A$ if and only if *there exists* a winning strategy of player \mathbf{II} in $G(t, h, \eta)$ where (t, h) is a tree that codes A (here one needs the assumption $\kappa^{<\kappa} = \kappa$ to be able to code the strategies into the elements of κ^κ). By Remark 16 above if A is Borel, then there is also such a tree. Since $\text{Borel} \subset \text{Borel}^*$ by Remark 16 and Borel is closed under taking complements, Borel sets are Δ_1^1 .

The fact that Δ_1^1 sets are Borel^* is a more complicated issue; it follows from a separation theorem proved in [22]. The separation theorem says that any two disjoint Σ_1^1 sets can be separated by Borel^* sets. It is proved in [22] for $\kappa = \omega_1$, but the proof generalizes to any κ (with $\kappa^{<\kappa} = \kappa$). \square

Additionally we have the following results:

18. **Theorem.** (1) $\text{Borel} \subsetneq \Delta_1^1$.

(2) $\Delta_1^1 \subsetneq \Sigma_1^1$.

(3) If $V = L$, then $\text{Borel}^* = \Sigma_1^1$.

(4) It is consistent that $\Delta_1^1 \subsetneq \text{Borel}^*$.

Proof. (Sketch)

(1) The following universal Borel set is not Borel itself, but is Δ_1^1 :

$$B = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid \eta \text{ is in the set coded by } (t_\xi, h_\xi)\},$$

where $\xi \mapsto (t_\xi, h_\xi)$ is a continuous coding of $(\kappa^+\omega\text{-tree, labeling})$ -pairs in such a way that for all $\kappa^+\omega$ -trees $t \subset \kappa^{<\omega}$ and labelings h there is ξ with $(t_\xi, h_\xi) = (t, h)$. It is not Borel since if it were, then the diagonal's complement

$$D = \{\eta \mid (\eta, \eta) \notin B\}$$

would be a Borel set which it is not, since it cannot be coded by any (t_ξ, h_ξ) . On the other hand its complement $C = (2^\kappa)^2 \setminus B$ is Σ_1^1 , because $(\eta, \xi) \in C$ if and only if *there exists* a winning strategy of player **I** in the Borel-game $G(t_\xi, h_\xi, \eta)$ and the latter can be coded to a Borel set. It is left to the reader to verify that when $\kappa > \omega$, then the set

$$F = \{(\eta, \xi, \nu) \mid \nu \text{ codes a w.s. for } \mathbf{I} \text{ in } G(t_\xi, h_\xi, \eta)\}$$

is closed.

The existence of an isomorphism relation which is Δ_1^1 but not Borel follows from Theorems 70 and 71.

- (2) Similarly as above (and similarly as in the case $\kappa = \omega$), take a universal Σ_1^1 -set $A \subset 2^\kappa \times 2^\kappa$ with the property that if $B \subset 2^\kappa$ is any Σ_1^1 -set, then there is $\eta \in 2^\kappa$ such that $B \times \{\eta\} \subset A$. This set can be constructed as in the case $\kappa = \omega$, see [15]. The diagonal $\{\eta \mid (\eta, \eta) \in A\}$ is Σ_1^1 but not Π_1^1 .
- (3) Suppose $V = L$ and $A \subset 2^\kappa$ is Σ_1^1 . There exists a formula $\varphi(x, \xi)$ with parameter $\xi \in 2^\kappa$ which is Σ_1 in the Levy hierarchy (see [15]) and for all $\eta \in 2^\kappa$ we have

$$\eta \in A \iff L \models \varphi(\eta, \xi).$$

Now we have that $\eta \in A$ if and only if the set

$$\{\alpha < \kappa \mid \exists \beta (\eta \restriction \alpha, \xi \restriction \alpha \in L_\beta, L_\beta \models (\text{ZF}^- \wedge (\alpha \text{ is a cardinal}) \wedge \varphi(\eta \restriction \alpha, \xi \restriction \alpha)))\}$$

contains an ω -cub set.

But the ω -cub filter is Borel* so A is also Borel*.

- (4) This follows from the clauses (1), (6) and (7) of Theorem 49 below. □

Open Problem. Is it consistent that Borel* is a proper subclass of Σ_1^1 , or even equals Δ_1^1 ? Is it consistent that all the inclusions are proper at the same time: $\Delta_1^1 \subsetneq \text{Borel}^* \subsetneq \Sigma_1^1$?

19. Theorem. *For a set $S \subset \kappa^\kappa$ the following are equivalent.*

- (1) S is Σ_1^1 ,
- (2) S is a projection of a Borel set,

- (3) S is a projection of a Σ_1^1 set,
 (4) S is a continuous image of a closed set.

Proof. Let us go in the order.

- (1) \Rightarrow (2): Closed sets are Borel.
 (2) \Rightarrow (3): The same proof as in the standard case $\kappa = \omega$ gives that Borel sets are Σ_1^1 (see for instance [15]).
 (3) \Rightarrow (4): Let $A \subset \kappa^\kappa \times \kappa^\kappa$ be a Σ_1^1 set which is the projection of A , $S = \text{pr}_0 A$. Then let $C \subset \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ be a closed set such that $\text{pr}_1 C = A$. Here $\text{pr}_0: \kappa^\kappa \times \kappa^\kappa \rightarrow \kappa^\kappa$ and $\text{pr}_1: \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa \rightarrow \kappa^\kappa \times \kappa^\kappa$ are the obvious projections. Let $f: \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa \rightarrow \kappa^\kappa$ be a homeomorphism. Then S is the image of the closed set $f[C]$ under the continuous map $\text{pr}_0 \circ \text{pr}_1 \circ f^{-1}$.
 (4) \Rightarrow (1): The image of a closed set under a continuous map f is the projection of the graph of f restricted to that closed set. It is a basic topological fact that a graph of a continuous partial function with closed domain is closed (provided the range is Hausdorff). \square

20. **Theorem** ([22]). *Borel* sets are closed under unions and intersections of size κ .* \square

21. **Definition.** A Borel* set B is *determined* if there exists a tree t and a labeling function h such that the corresponding game $G(t, h, \eta)$ is determined for all $\eta \in \kappa^\kappa$ and

$$B = \{\eta \mid \mathbf{II} \text{ has a winning strategy in } G(t, h, \eta)\}.$$

22. **Theorem** ([22]). *Δ_1^1 sets are exactly the determined Borel* sets.* \square

3. BOREL SETS, Δ_1^1 SETS AND INFINITARY LOGIC

3.1. **The Language $L_{\kappa+\kappa}$ and Borel Sets.** The interest in the class of Borel sets is explained by the fact that the Borel sets are relatively simple yet at the same time this class includes many interesting definable sets. We prove Vaught's theorem (Theorem 24), which equates “invariant” Borel sets with those definable in the infinitary language $L_{\kappa+\kappa}$. Recall that models \mathcal{A} and \mathcal{B} of size κ are $L_{\kappa+\kappa}$ -equivalent if and only if they are $L_{\infty\kappa}$ -equivalent. Vaught proved his theorem for the case $\kappa = \omega_1$ assuming CH in [36], but the proof works for arbitrary κ assuming $\kappa^{<\kappa} = \kappa$.

23. Definition. Denote by S_κ the set of all permutations of κ . If $u \in \kappa^{<\kappa}$, denote

$$\bar{u} = \{p \in S_\kappa \mid p^{-1} \upharpoonright \text{dom } u = u\}.$$

Note that $\bar{\emptyset} = S_\kappa$ and if $u \in \kappa^\alpha$ is not injective, then $\bar{u} = \emptyset$.

A permutation $p: \kappa \rightarrow \kappa$ acts on 2^κ by

$$p\eta = \xi \iff p: \mathcal{A}_\eta \rightarrow \mathcal{A}_\xi \text{ is an isomorphism.}$$

The map $\eta \mapsto p\eta$ is well defined for every p and it is easy to check that it defines an action of the permutation group S_κ on the space 2^κ . We say that a set $A \subset 2^\kappa$ is *closed under permutations* if it is a union of orbits of this action.

24. Theorem ([36], $\kappa^{<\kappa} = \kappa$). *A set $B \subset \kappa^\kappa$ is Borel and closed under permutations if and only if there is a sentence φ in $L_{\kappa+\kappa}$ such that $B = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$.*

Proof. Let φ be a sentence in $L_{\kappa+\kappa}$. Then $\{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models \varphi\}$ is closed under permutations, because if $\eta = p\xi$, then $\mathcal{A}_\eta \cong \mathcal{A}_\xi$ and $\mathcal{A}_\eta \models \varphi \iff \mathcal{A}_\xi \models \varphi$ for every sentence φ . If φ is a formula with parameters $(a_i)_{i<\alpha} \in \kappa^\alpha$, one easily verifies by induction on the complexity of φ that the set

$$\{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models \varphi((a_i)_{i<\alpha})\}$$

is Borel. This of course implies that for every sentence φ the set $\{\eta \mid \mathcal{A}_\eta \models \varphi\}$ is Borel.

The converse is less trivial. Note that the set of permutations $S_\kappa \subset \kappa^\kappa$ is Borel, since

$$S_\kappa = \bigcap_{\beta < \kappa} \bigcup_{\alpha < \kappa} \underbrace{\{\eta \mid \eta(\alpha) = \beta\}}_{\text{open}} \cap \bigcap_{\alpha < \beta < \kappa} \underbrace{\{\eta \mid \eta(\alpha) \neq \eta(\beta)\}}_{\text{open}}. \quad (\cdot)$$

For a set $A \subset \kappa^\kappa$ and $u \in \kappa^{<\kappa}$, define

$$A^{*u} = \{\eta \in 2^\kappa \mid \{p \in \bar{u} \mid p\eta \in A\} \text{ is co-meager in } \bar{u}\}.$$

From now on in this section we will write “ $\{p \in \bar{u} \mid p\eta \in A\}$ is co-meager”, when we really mean “co-meager in \bar{u} ”.

Let us show that the set

$$Z = \{A \subset 2^\kappa \mid A \text{ is Borel, } A^{*u} \text{ is } L_{\kappa+\kappa}\text{-definable for all } u \in \kappa^{<\kappa}\}$$

contains all the basic open sets, is closed under intersections of size κ and under complementation in the three steps (a), (b) and (c) below. This implies that Z is the collection of all Borel sets. We will additionally keep track of the fact that the formula, which defines A^{*u} depends only on A and $\text{dom } u$, i.e. for each

$\beta < \kappa$ and Borel set A there exists $\varphi = \varphi_\beta^A$ such that for all $u \in \kappa^\beta$ we have $A^{*u} = \{\eta \mid \mathcal{A}_\eta \models \varphi((u_i)_{i < \beta})\}$. Setting $u = \emptyset$, we have the intended result, because $A^{*\emptyset} = A$ for all A which are closed under permutations and φ is a sentence (with no parameters).

If A is fixed we denote $\varphi_\beta^A = \varphi_\beta$.

- (a) Assume $q \in 2^{<\kappa}$ and let N_q be the corresponding basic open set. Let us show that $N_q \in Z$. Let $u \in \kappa^\beta$ be arbitrary. We have to find $\varphi_\beta^{N_q}$. Let θ be a quantifier free formula with α parameters such that:

$$N_q = \{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models \theta((\gamma)_{\gamma < \alpha})\}.$$

Here $(\gamma)_{\gamma < \alpha}$ denotes both an initial segment of κ as well as an α -tuple of the structure. Suppose $\alpha \leq \beta$. We have $p \in \bar{u} \Rightarrow u \subset p^{-1}$, so

$$\begin{aligned} \eta \in N_q^{*u} &\iff \{p \in \bar{u} \mid p\eta \in N_q\} \text{ is co-meager} \\ &\iff \{p \in \bar{u} \mid \mathcal{A}_{p\eta} \models \theta((\gamma)_{\gamma < \alpha})\} \text{ is co-meager} \\ &\iff \{p \in \bar{u} \mid \mathcal{A}_\eta \models \theta((p^{-1}(\gamma))_{\gamma < \alpha})\} \text{ is co-meager} \\ &\iff \{p \in \bar{u} \mid \underbrace{\mathcal{A}_\eta \models \theta((u_\gamma)_{\gamma < \alpha})}_{\text{independent of } p}\} \text{ is co-meager} \\ &\iff \mathcal{A}_\eta \models \theta((u_\gamma)_{\gamma < \alpha}). \end{aligned}$$

Then $\varphi_\beta = \theta$.

Assume then that $\alpha > \beta$. By the above, we still have

$$\eta \in N_q^{*u} \iff E = \{p \in \bar{u} \mid \mathcal{A}_\eta \models \theta((p^{-1}(\gamma))_{\gamma < \alpha})\} \text{ is co-meager}.$$

Assume that $w = (w_\gamma)_{\gamma < \alpha} \in \kappa^\alpha$ is an arbitrary sequence with no repetition and such that $u \subset w$. Since \bar{w} is an open subset of \bar{u} and E is co-meager, there is $p \in \bar{w} \cap E$. Because $p \in E$, we have $\mathcal{A}_\eta \models \theta((p^{-1}(\gamma))_{\gamma < \alpha})$. On the other hand $p \in \bar{w}$, so we have $w \subset p^{-1}$, i.e. $w_\gamma = w(\gamma) = p^{-1}(\gamma)$ for $\gamma < \alpha$. Hence

$$\mathcal{A}_\eta \models \theta((w_\gamma)_{\gamma < \alpha}). \quad (\star)$$

On the other hand, if for every injective $w \in \kappa^\alpha$, $w \supset u$, we have (\star) , then in fact $E = \bar{u}$ and is trivially co-meager. Therefore we have an equivalence:

$$\eta \in N_q^{*u} \iff (\forall w \supset u)(w \in \kappa^\alpha \wedge w \text{ inj.} \Rightarrow \mathcal{A}_\eta \models \theta((w_\gamma)_{\gamma < \alpha})).$$

But the latter can be expressed in the language $L_{\kappa+k}$ by the formula $\varphi_\beta((w_i)_{i<\beta})$:

$$\bigwedge_{i<j<\beta} (w_i \neq w_j) \wedge \left(\bigvee_{\beta \leq i < \alpha} w_i \right) \left(\bigwedge_{i<j<\alpha} (w_i \neq w_j) \rightarrow \theta((w_i)_{i<\alpha}) \right)$$

θ was defined to be a formula defining N_q with parameters. It is clear thus that θ is independent of u . Furthermore the formulas constructed above from θ depend only on $\beta = \text{dom } u$ and on θ . Hence the formulas defining N_q^{*u} and N_q^{*v} for $\text{dom } u = \text{dom } v$ are the same modulo parameters.

- (b) For each $i < \kappa$ let $A_i \in Z$. We want to show that $\bigcap_{i<\kappa} A_i \in Z$. Assume that $u \in \kappa^{<\kappa}$ is arbitrary. It suffices to show that

$$\bigcap_{i<\kappa} (A_i^{*u}) = \left(\bigcap_{i<\kappa} A_i \right)^{*u},$$

because then $\varphi_\beta^{\bigcap_{i<\kappa} A_i}$ is just the κ -conjunction of the formulas $\varphi_\beta^{A_i}$ which exist by the induction hypothesis. Clearly the resulting formula depends again only on $\text{dom } u$ if the previous did. Note that a κ -intersection of co-meager sets is co-meager. Now

$$\begin{aligned} \eta \in \bigcap_{i<\kappa} (A_i^{*u}) &\iff (\forall i < \kappa) (\{p \in \bar{u} \mid p\eta \in A_i\} \text{ is co-meager}) \\ &\iff (\forall i < \kappa) (\forall i < \kappa) (\{p \in \bar{u} \mid p\eta \in A_i\} \text{ is co-meager}) \\ &\iff \bigcap_{i<\kappa} \{p \in \bar{u} \mid p\eta \in A_i\} \text{ is co-meager} \\ &\iff \{p \in \bar{u} \mid p\eta \in \bigcap_{i<\kappa} A_i\} \text{ is co-meager} \\ &\iff \eta \in \left(\bigcap_{i<\kappa} A_i \right)^{*u}. \end{aligned}$$

- (c) Assume that $A \in Z$ i.e. that A^{*u} is definable for any u . Let $\varphi_{\text{dom } u}$ be the formula, which defines A^{*u} . Let now $u \in \kappa^{<\kappa}$ be arbitrary and let us show that $(A^c)^{*u}$ is definable. We will show that

$$(A^c)^{*u} = \bigcap_{v \supset u} (A^{*v})^c$$

i.e. for all η

$$(1) \quad \eta \in (A^c)^{*u} \iff \forall v \supset u (\eta \notin A^{*v}).$$

Granted this, one can write the formula “ $\forall v \supset u \neg \varphi_{\text{dom } u}((v_i)_{i < \text{dom } v})$ ”, which is not of course the real $\varphi_\beta^{A^c}$ which we will write in the end of the proof.

To prove (1) we have to show first that for all $\eta \in \kappa^\kappa$ the set $B = \{p \in \bar{u} \mid p\eta \in A\}$ has the Property of Baire (P.B.), see Section 4.3.

The set of all permutations $S_\kappa \subset \kappa^\kappa$ is Borel by (\cdot) on page 18. The set \bar{u} is an intersection of S_κ with an open set. Again the set $\{p \in \bar{u} \mid p\eta \in A\}$ is the intersection of \bar{u} and the inverse image of A under the continuous map $(p \mapsto p\eta)$, so is Borel and so has the Property of Baire.

We can now turn to proving the equivalence (1). First “ \Leftarrow ”:

$$\eta \notin (A^c)^{*u}$$

$$\implies B = \{p \in \bar{u} \mid p\eta \in A\} \text{ is not meager in } \bar{u}$$

$$\implies \text{By P.B. of } B \text{ there is a non-empty open } U \text{ such that } U \setminus B \text{ is meager}$$

$$\implies \text{There is non-empty } \bar{v} \subset \bar{u} \text{ such that } \bar{v} \setminus B \text{ is meager.}$$

$$\implies \text{There exists } \bar{v} \subset \bar{u} \text{ such that } \{p \in \bar{v} \mid p\eta \in A\} = \bar{v} \cap B \text{ is co-meager}$$

$$\implies \exists v \supset u (\eta \in A^{*v}).$$

And then the other direction “ \Rightarrow ”:

$$\eta \in (A^c)^{*u} \implies \{p \in \bar{u} \mid p\eta \in A\} \text{ is meager}$$

$$\implies \text{for all } \bar{v} \subset \bar{u} \text{ the set } \{p \in \bar{v} \mid p\eta \in A\} \text{ is meager.}$$

$$\implies \forall \bar{v} \subset \bar{u} (\eta \notin A^{*v}).$$

Let us now write the formula $\psi = \varphi_\beta^{A^c}$ such that

$$\forall \bar{v} \subset \bar{u} (\eta \notin A^{*v}) \iff \mathcal{A}_\eta \models \psi((u_i)_{i < \beta}),$$

where $\beta = \text{dom } u$: let $\psi((u_i)_{i < \beta})$ be

$$\bigwedge_{\beta \leq \gamma < \kappa} \bigvee_{i < \gamma} x_i \left(\left[\bigwedge_{j < \beta} (x_j = u_j) \wedge \bigwedge_{i < j < \gamma} (x_i \neq x_j) \right] \rightarrow \neg \varphi_\gamma((x_i)_{i < \gamma}) \right).$$

One can easily see, that this is equivalent to $\forall v \supset u (\neg \varphi_{\text{dom } v}((v_i)_{i < \text{dom } v}))$ and that ψ depends only on $\text{dom } u$ modulo parameters. \square

25. *Remark.* If $\kappa^{<\kappa} > \kappa$, then the direction from right to left of the above theorem does not in general hold. Let $\langle \kappa, \leq, A \rangle$ be a model with domain κ , $A \subset \kappa$ and \leq a well ordering of κ of order type κ . Väänänen and Shelah have shown in [30]

(Corollary 17) that if $\kappa = \lambda^+$, $\kappa^{<\kappa} > \kappa$, $\lambda^{<\lambda} = \lambda$ and a forcing axiom holds (and $\omega_1^L = \omega_1$ if $\lambda = \omega$) then there is a sentence of $L_{\kappa\kappa}$ defining the set

$$\text{STAT} = \{\langle \kappa, \leq, A \rangle \mid A \text{ is stationary}\}.$$

If now STAT is Borel, then so would be the set CUB defined in Section 4.3, but by Theorem 49 this set cannot be Borel since Borel sets have the Property of Baire by Theorem 45.

Open Problem. Does the direction left to right of Theorem 24 hold without the assumption $\kappa^{<\kappa} = \kappa$?

3.2. The Language $M_{\kappa+\kappa}$ and Δ_1^1 -Sets. In this section we will present a theorem similar to Theorem 24. It is also a generalization of the known result which follows from [22] and [34]:

26. Theorem ([22, 34]:). *Let \mathcal{A} be a model of size ω_1 . Then the isomorphism type $I = \{\eta \mid \mathcal{A}_\eta \cong \mathcal{A}\}$ is Δ_1^1 if and only if there is a sentence φ in $M_{\kappa+\kappa}$ such that $I = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ and $2^\kappa \setminus I = \{\eta \mid \mathcal{A}_\eta \models \sim \varphi\}$, where $\sim \theta$ is the dual of θ .*

The idea of the proof of the following Theorem is due to Sam Coskey and Philipp Schlicht:

27. Theorem ($\kappa^{<\kappa} = \kappa$). *A set $D \subset 2^\kappa$ is Δ_1^1 and closed under permutations if and only if there is a sentence φ in $M_{\kappa+\kappa}$ such that $D = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ and $\kappa^\kappa \setminus D = \{\eta \mid \mathcal{A}_\eta \models \sim \varphi\}$, where $\sim \theta$ is the dual of θ .*

We have to define these concepts before the proof.

28. Definition (Karttunen [17]). Let λ and κ be cardinals. The language $M_{\lambda\kappa}$ is then defined to be the set of pairs (t, \mathcal{L}) of a tree t and a labeling function \mathcal{L} . The tree t is a $\lambda\kappa$ -tree where the limits of increasing sequences of t exist and are unique. The labeling \mathcal{L} is a function satisfying the following conditions:

- (1) $\mathcal{L}: t \rightarrow a \cup \bar{a} \cup \{\wedge, \vee\} \cup \{\exists x_i \mid i < \kappa\} \cup \{\forall x_i \mid i < \kappa\}$ where a is the set of atomic formulas and \bar{a} is the set of negated atomic formulas.
- (2) If $x \in t$ has no successors, then $\mathcal{L}(t) \in a \cup \bar{a}$.
- (3) If $x \in t$ has exactly one immediate successor then $\mathcal{L}(t)$ is either $\exists x_i$ or $\forall x_i$ for some $i < \kappa$.
- (4) Otherwise $\mathcal{L}(t) \in \{\vee, \wedge\}$.
- (5) If $x < y$, $\mathcal{L}(x) \in \{\exists x_i, \forall x_i\}$ and $\mathcal{L}(y) \in \{\exists x_j, \forall x_j\}$, then $i \neq j$.

29. Definition. Truth for $M_{\lambda\kappa}$ is defined in terms of a semantic game. Let (t, \mathcal{L}) be the pair which corresponds to a particular sentence φ and let \mathcal{A} be a model. The semantic game $S(\varphi, \mathcal{A}) = S(t, \mathcal{L}, \mathcal{A})$ for $M_{\lambda\kappa}$ is played by players **I** and **II** as follows. At the first move the players are at the root and later in the game at some other element of t . Let us suppose that they are at the element $x \in t$. If $\mathcal{L}(x) = \bigvee$, then Player **II** chooses a successor of x and the players move to that chosen element. If $\mathcal{L}(x) = \bigwedge$, then player **I** chooses a successor of x and the players move to that chosen element. If $\mathcal{L}(x) = \forall x_i$ then player **I** picks an element $a_i \in \mathcal{A}$ and if $\mathcal{L}(x) = \exists x_i$ then player **II** picks an element a_i and they move to the immediate successor of x . If they come to a limit, they move to the unique supremum. If x is a maximal element of t , then they plug the elements a_i in place of the corresponding free variables in the atomic formula $\mathcal{L}(x)$. Player **II** wins if this atomic formula is true in \mathcal{A} with these interpretations. Otherwise player **I** wins.

We define $\mathcal{A} \models \varphi$ if and only if **II** has a winning strategy in the semantic game.

Given a sentence φ , the sentence $\sim \varphi$ is defined by modifying the labeling function as follows. The atomic formulas are replaced by their negations, the symbols \bigvee and \bigwedge switch places and the quantifiers \forall and \exists switch places. A sentence $\varphi \in M_{\lambda\kappa}$ is *determined* if for all models \mathcal{A} either $\mathcal{A} \models \varphi$ or $\mathcal{A} \models \sim \varphi$.

Now the statement of Theorem 27 makes sense. Theorem 27 concerns a sentence φ whose dual defines the complement of the set defined by φ among the models of size κ , so it is determined in that model class. Before the proof let us recall a separation theorem for $M_{\kappa+\kappa}$, Theorem 3.9 from [32]:

30. Theorem. Assume $\kappa^{<\kappa} = \lambda$ and let $\exists R\varphi$ and $\exists S\psi$ be two Σ_1^1 sentences where φ and ψ are in $M_{\kappa+\kappa}$ and $\exists R$ and $\exists S$ are second order quantifiers. If $\exists R\varphi \wedge \exists S\psi$ does not have a model, then there is a sentence $\theta \in M_{\lambda+\lambda}$ such that for all models \mathcal{A}

$$\mathcal{A} \models \exists R\varphi \Rightarrow \mathcal{A} \models \theta \text{ and } \mathcal{A} \models \exists S\psi \Rightarrow \mathcal{A} \models \sim \theta. \quad \square$$

31. Definition. For a tree t , let σt be the tree of downward closed linear subsets of t ordered by inclusion.

Proof of Theorem 27. Let us first show that if φ is an arbitrary sentence of $M_{\kappa+\kappa}$, then $D_\varphi = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ is Σ_1^1 . The proof has the same idea as the proof of Theorem 17 that $\text{Borel}^* \subset \Sigma_1^1$. Note that this implies that if $\sim \varphi$ defines the complement of D_φ in 2^κ , then D_φ is Δ_1^1 .

A strategy in the semantic game $S(\varphi, \mathcal{A}_\eta) = S(t, \mathcal{L}, \mathcal{A}_\eta)$ is a function

$$v: \sigma t \times (\text{dom } \mathcal{A}_\eta)^{<\kappa} \rightarrow t \cup (t \times \text{dom } \mathcal{A}_\eta).$$

This is because the previous moves always form an initial segment of a branch of the tree together with the sequence of constants picked by the players from $\text{dom } \mathcal{A}_\eta$ at the quantifier moves, and a move consists either of going to some node of the tree or going to a node of the tree together with choosing an element from $\text{dom } \mathcal{A}_\eta$. By the convention that $\text{dom } \mathcal{A}_\eta = \kappa$, a strategy becomes a function

$$v: \sigma t \times \kappa^{<\kappa} \rightarrow t \cup (t \times \kappa).$$

Because t is a $\kappa^+ \kappa$ -tree, there are fewer than κ moves in a play (there are no branches of length κ and the players go up the tree on each move). Let

$$f: \sigma t \times \kappa^{<\kappa} \rightarrow \kappa$$

be any bijection and let

$$g: t \cup (t \times \kappa) \rightarrow \kappa$$

be another bijection. Let F be the bijection

$$F: (t \cup (t \times \kappa))^{\sigma t \times \kappa^{<\kappa}} \rightarrow \kappa^\kappa$$

defined by $F(v) = g \circ v \circ f^{-1}$. Let

$$C = \{(\eta, \xi) \mid F^{-1}(\xi) \text{ is a winning strategy of } \mathbf{II} \text{ in } S(t, \mathcal{L}, \mathcal{A}_\eta)\}.$$

Clearly D_φ is the projection of C . Let us show that C is closed. Consider an element (η, ξ) in the complement of C . We shall show that there is an open neighbourhood of (η, ξ) outside C . Denote $v = F^{-1}(\xi)$. Since v is not a winning strategy there is a strategy τ of \mathbf{I} that beats v . There are $\alpha + 1 < \kappa$ moves in the play $\tau * v$ (by definition all branches have successor order type). Assume that $b = (x_i)_{i \leq \alpha}$ is the chosen branch of the tree and $(c_i)_{i < \alpha}$ the constants picked by the players. Let $\beta < \kappa$ be an ordinal with the properties $\{f((x_i)_{i < \gamma}, (c_i)_{i < \gamma}) \mid \gamma \leq \alpha + 1\} \subset \beta$ and

$$\eta' \in N_{\eta \upharpoonright \beta} \rightarrow \mathcal{A}_{\eta'} \not\models \mathcal{L}(x_\alpha)((c_i)_{i < \alpha}). \quad (\star)$$

Such β exists, since $|\{f((x_i)_{i < \gamma}, (c_i)_{i < \gamma}) \mid \gamma \leq \alpha + 1\}| < \kappa$ and $\mathcal{L}(x_\alpha)$ is a (possibly negated) atomic formula which is not true in \mathcal{A}_η , because \mathbf{II} lost the game $\tau * v$ and because already a fragment of size $< \kappa$ of \mathcal{A}_η decides this. Now

if $(\eta', \xi') \in N_{\eta \upharpoonright \beta} \times N_{\xi \upharpoonright \beta}$ and $v' = F^{-1}(\xi')$, then $v * \tau$ is the same play as $\tau * v'$. So $\mathcal{A}_{\eta'} \not\models \mathcal{L}(x_\alpha)((c_i)_{i < \alpha})$ by (\star) and (η', ξ') is not in C and

$$N_{\eta \upharpoonright \beta} \times N_{\xi \upharpoonright \beta}$$

is the intended open neighbourhood of (η, ξ) outside C . This completes the “if”-part of the proof.

Now for a given $A \in \Delta_1^1$ which is closed under permutations we want to find a sentence $\varphi \in M_{\kappa+\kappa}$ such that $A = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ and $2^\kappa \setminus A = \{\eta \mid \mathcal{A}_\eta \models \sim \varphi\}$. By our assumption $\kappa^{<\kappa} = \kappa$ and Theorems 22 and 30, it is enough to show that for a given Borel* set B which is closed under permutations, there is a sentence $\exists R\psi$ which is Σ_1^1 over $M_{\kappa+\kappa}$ (as in the formulation of Theorem 30), such that $B = \{\eta \mid \mathcal{A}_\eta \models \exists R\psi\}$.

The sentence “ R is a well ordering of the universe of order type κ ”, is definable by the formula $\theta = \theta(R)$ of $L_{\kappa+\kappa} \subset M_{\kappa+\kappa}$:

“ R is a linear ordering on the universe”

$$(2) \quad \begin{aligned} & \wedge \left(\bigvee_{i < \omega} x_i \right) \left(\bigvee_{i < \omega} \neg R(x_{i+1}, x_i) \right) \\ & \wedge \forall x \bigvee_{\alpha < \kappa} \bigwedge_{i < \alpha} y_i \left[\left(\forall y (R(y, x) \rightarrow \bigvee_{i < \alpha} y_i = y) \right) \right]. \end{aligned}$$

(We assume $\kappa > \omega$, so the infinite quantification is allowed. The second row says that there are no descending sequences of length ω and the third row says that the initial segments are of size less than κ . This ensures that $\theta(R)$ says that R is a well ordering of order type κ).

Let t and h be the tree and the labeling function corresponding to B . Define the tree t^* as follows.

- (1) Assume that b is a branch of t with $h(b) = N_{\xi \upharpoonright \alpha}$ for some $\xi \in \kappa^\kappa$ and $\alpha < \kappa$. Then attach a sequence of order type α^* on top of b where

$$\alpha^* = \bigcup_{s \in \pi^{-1}[\alpha]} \text{ran } s,$$

where π is the bijection $\kappa^{<\omega} \rightarrow \kappa$ used in the coding, see Definition 12 on page 12.

- (2) Do this to each branch of t and add a root r to the resulting tree.

After doing this, the resulting tree is t^* . Clearly it is a $\kappa^+\kappa$ -tree, because t is. Next, define the labeling function \mathcal{L} . If $x \in t$ then either $\mathcal{L}(x) = \bigwedge$ or $\mathcal{L}(x) = \bigvee$ depending on whether it is player **I**'s move or player **II**'s move: formally let $n < \omega$ be such that $\text{OTP}(\{y \in t^* \mid y \leq x\}) = \alpha + n$ where α is a limit ordinal or 0; then if n is odd, put $\mathcal{L}(x) = \bigwedge$ and otherwise $\mathcal{L}(x) = \bigvee$. If $x = r$ is the root, then $\mathcal{L}(x) = \bigwedge$. Otherwise, if x is not maximal, define

$$\beta = \text{OTP}\{y \in t^* \setminus (t \cup \{r\}) \mid y \leq x\}$$

and set $\mathcal{L}(x) = \exists x_\beta$.

Next we will define the labeling of the maximal nodes of t^* . By definition these should be atomic formulas or negated atomic formulas, but it is clear that they can be replaced without loss of generality by any formula of $M_{\kappa+\kappa}$; this fact will make the proof simpler. Assume that x is maximal in t^* . $\mathcal{L}(x)$ will depend only on $h(b)$ where b is the unique branch of t leading to x . Let us define $\mathcal{L}(x)$ to be the formula of the form $\theta \wedge \Theta_b((x_i)_{i < \alpha^*})$, where θ is defined above and Θ_b is defined below. The idea is that

$$\mathcal{A}_\eta \models \Theta_b((a_\gamma)_{\gamma < \alpha^*}) \iff \eta \in h(b) \text{ and } \forall \gamma < \alpha^* (a_\gamma = \gamma).$$

Let us define such a Θ_b . Suppose that ξ and α are such that $h(b) = N_{\xi \upharpoonright \alpha}$. Define for $s \in \pi^{-1}[\alpha]$ the formula A_b^s as follows:

$$A_b^s = \begin{cases} P_{\text{dom } s}, & \text{if } \mathcal{A}_\xi \models P_{\text{dom } s}((s(i))_{i \in \text{dom } s}) \\ \neg P_{\text{dom } s}, & \text{if } \mathcal{A}_\xi \not\models P_{\text{dom } s}((s(i))_{i \in \text{dom } s}). \end{cases}$$

Then define

$$\begin{aligned} \psi_0((x_i)_{i < \alpha^*}) &= \bigwedge_{i < \alpha^*} \left[\forall y (R(y, x_i) \leftrightarrow \bigvee_{j < i} (y = x_j)) \right] \\ \psi_1((x_i)_{i < \alpha^*}) &= \bigwedge_{s \in \pi^{-1}[\alpha]} A_b^s((x_{s(i)})_{i \in \text{dom } s}), \\ \Theta_b &= \psi_0 \wedge \psi_1. \end{aligned}$$

The disjunction over the empty set is considered false.

Claim 1. Suppose for all η , R is the standard order relation on κ . Then

$$(\mathcal{A}_\eta, R) \models \Theta_b((a_\gamma)_{\gamma < \alpha^*}) \iff \eta \in h(b) \wedge \forall \gamma < \alpha^* (\alpha_\gamma = \gamma).$$

Proof of Claim 1. Suppose $\mathcal{A}_\eta \models \Theta_b((a_\gamma)_{\gamma < \alpha^*})$. Then by $\mathcal{A}_\eta \models \psi_0((a_\gamma)_{\gamma < \alpha^*})$ we have that $(a_\gamma)_{\gamma < \alpha^*}$ is an initial segment of $\text{dom } \mathcal{A}_\eta$ with respect to R . But

$(\text{dom } \mathcal{A}_\eta, R) = (\kappa, <)$, so $\forall \gamma < \alpha^* (\alpha_\gamma = \gamma)$. Assume that $\beta < \alpha$ and $\eta(\beta) = 1$ and denote $s = \pi^{-1}(\beta)$. Then $\mathcal{A}_\eta \models P_{\text{dom } s}((s(i))_{i \in \text{dom } s})$. Since Θ is true in \mathcal{A}_η as well, we must have $A_b^s = P_{\text{dom } s}$ which by definition means that $\mathcal{A}_\xi \models P_{\text{dom } s}((s(i))_{i \in \text{dom } s})$ and hence $\xi(\beta) = \xi(\pi(s)) = 1$. In the same way one shows that if $\eta(\beta) = 0$, then $\xi(\beta) = 0$ for all $\beta < \alpha$. Hence $\eta \upharpoonright \alpha = \xi \upharpoonright \alpha$.

Assume then that $a_\gamma = \gamma$ for all $\gamma < \alpha^*$ and that $\eta \in N_{\xi \upharpoonright \alpha}$. Then \mathcal{A}_η trivially satisfies ψ_0 . Suppose that $s \in \pi^{-1}[\alpha]$ is such that $\mathcal{A}_\xi \models P_{\text{dom } s}((s(i))_{i \in \text{dom } s})$. Then $\xi(\pi(s)) = 1$ and since $\pi(s) < \alpha$, also $\eta(\pi(s)) = 1$, so $\mathcal{A}_\eta \models P_{\text{dom } s}((s(i))_{i \in \text{dom } s})$. Similarly one shows that if

$$\mathcal{A}_\xi \not\models P_{\text{dom } s}((s(i))_{i \in \text{dom } s}),$$

then $\mathcal{A}_\eta \not\models P_{\text{dom } s}((s(i))_{i \in \text{dom } s})$. This shows that $\mathcal{A}_\eta \models A_b^s((s(i))_{i \in \text{dom } s})$ for all s . Hence \mathcal{A}_η satisfies ψ_1 , so we have $\mathcal{A}_\eta \models \Theta$. \square Claim 1

Claim 2. t, h, t^* and \mathcal{L} are such that for all $\eta \in \kappa^\kappa$

$$\mathbf{II} \upharpoonright G(t, h, \eta) \iff \exists R \subset (\text{dom } \mathcal{A}_\eta)^2 \quad \mathbf{II} \upharpoonright S(t^*, \mathcal{L}, \mathcal{A}_\eta).$$

Proof of Claim 2. Suppose σ is a winning strategy of \mathbf{II} in $G(t, h, \eta)$. Let R be the well ordering of $\text{dom } \mathcal{A}_\eta$ such that $(\text{dom } \mathcal{A}_\eta, R) = (\kappa, <)$. Consider the game $S(t^*, \mathcal{L}, \mathcal{A}_\eta)$. On the first move the players are at the root and player \mathbf{I} chooses where to go next. They go to to a minimal element of t . From here on \mathbf{II} uses σ as long as they are in t . Let us see what happens if they got to a maximal element of t , i.e. they picked a branch b from t . Since σ is a winning strategy of \mathbf{II} in $G(t, h, \eta)$, we have $\eta \in h(b)$ and $h(b) = N_{\xi \upharpoonright \alpha}$ for some ξ and α . For the next α moves the players climb up the tower defined in item (1) of the definition of t^* . All labels are of the form $\exists x_\beta$, so player \mathbf{II} has to pick constants from \mathcal{A}_η . She picks them as follows: for the variable x_β she picks $\beta \in \kappa = \text{dom } \mathcal{A}_\eta$. She wins now if $\mathcal{A}_\eta \models \Theta((\beta)_{\beta < \alpha^*})$ and $\mathcal{A}_\eta \models \theta$. But $\eta \in h(b)$, so by Claim 1 the former holds and the latter holds because we chose R to be a well ordering of order type κ .

Let us assume that there is no winning strategy of \mathbf{II} in $G(t, h, \eta)$. Let R be an arbitrary relation on $\text{dom } \mathcal{A}_\eta$. Here we shall finally use the fact that B is closed under permutations. Suppose R is not a well ordering of the universe of order type κ . Then after the players reached the final node of t^* , player \mathbf{I} chooses to go to θ and player \mathbf{II} loses. So we can assume that R is a well ordering of the

universe of order type κ . Let $p: \kappa \rightarrow \kappa$ be a bijection such that $p(\alpha)$ is the α^{th} element of κ with respect to R . Now p is a permutation and $\{\eta \mid \mathcal{A}_{p\eta} \in B\} = B$ since B is closed under permutations. So by our assumption that $\eta \notin B$ (i.e. $\mathbf{II} \nVdash G(t, h, \eta)$), we also have $p\eta \notin B$, i.e. player \mathbf{II} has no winning strategy in $G(t, h, p\eta)$ either.

Suppose σ is any strategy of \mathbf{II} in $S(t^*, \mathcal{L}, \mathcal{A}_\eta)$. Player \mathbf{I} imagines that σ is a strategy in $G(t, h, p\eta)$ and picks a strategy τ that beats it. In the game $S(t^*, \mathcal{L}, \mathcal{A}_\eta)$, as long as the players are still in t , player \mathbf{I} uses τ that would beat σ if they were playing $G(t, h, p\eta)$ instead of $S(t^*, \mathcal{L}, \eta)$. Suppose they picked a branch b of t . Now $p\eta \notin h(b)$. If \mathbf{II} wants to satisfy ψ_0 of the definition of Θ_b , she is forced to pick the constants $(a_i)_{i < \alpha^*}$ such that a_i is the i^{th} element of $\text{dom } \mathcal{A}_\eta$ with respect to R . Suppose that $\mathcal{A}_\eta \models \psi_1((a_i)_{i < \alpha^*})$ (recall $\Theta_b = \psi_0 \wedge \psi_1$). But then $\mathcal{A}_{p\eta} \models \psi_1((\gamma)_{\gamma < \alpha^*})$ and also $\mathcal{A}_{p\eta} \models \psi_0((\gamma)_{\gamma < \alpha^*})$, so by Claim 1 we should have $p\eta \in h(b)$ which is a contradiction. $\square_{\text{Claim 2}}$

$\square_{\text{Theorem 27}}$

4. GENERALIZATIONS FROM CLASSICAL DESCRIPTIVE SET THEORY

4.1. Simple Generalizations.

4.1.1. *The Identity Relation.* Denote by id the equivalence relation $\{(\eta, \xi) \in (2^\kappa)^2 \mid \eta = \xi\}$. With respect to our choice of topology, the natural generalization of the equivalence relation

$$E_0 = \{(\eta, \xi) \in 2^\omega \times 2^\omega \mid \exists n < \omega \forall m > n (\eta(m) = \xi(m))\}$$

is equivalence modulo sets of size $< \kappa$:

$$E_0^{<\kappa} = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid \exists \alpha < \kappa \forall \beta > \alpha (\eta(\beta) = \xi(\beta))\},$$

although the equivalences modulo sets of size $< \lambda$ for $\lambda < \kappa$ can also be studied:

$$E_0^{<\lambda} = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid \exists A \subset \kappa [|A| < \lambda \wedge \forall \beta \notin A (\eta(\beta) = \xi(\beta))] \},$$

but for $\lambda < \kappa$ these turn out to be bireducible with id (see below). Similarly one can define $E_0^{<\lambda}$ on κ^κ instead of 2^κ .

It makes no difference whether we define these relations on 2^κ or κ^κ since they become bireducible to each other:

32. Theorem. *Let $\lambda \leq \kappa$ be a cardinal and let $E_0^{<\lambda}(P)$ denote the equivalence relation $E_0^{<\lambda}$ on $P \in \{2^\kappa, \kappa^\kappa\}$ (notation defined above). Then*

$$E_0^{<\lambda}(2^\kappa) \leq_c E_0^{<\lambda}(\kappa^\kappa) \text{ and } E_0^{<\lambda}(\kappa^\kappa) \leq_c E_0^{<\lambda}(2^\kappa).$$

Note that when $\lambda = 1$, we have $E_0^{<1}(P) = \text{id}_P$.

Proof. In this proof we think of functions $\eta, \xi \in \kappa^\kappa$ as graphs $\eta = \{(\alpha, \eta(\alpha)) \mid \alpha < \kappa\}$. Fix a bijection $h: \kappa \rightarrow \kappa \times \kappa$. Let $f: 2^\kappa \rightarrow \kappa^\kappa$ be the inclusion, $f(\eta)(\alpha) = \eta(\alpha)$. Then f is easily seen to be a continuous reduction $E_0^{<\lambda}(2^\kappa) \leq_c E_0^{<\lambda}(\kappa^\kappa)$. Define $g: \kappa^\kappa \rightarrow 2^\kappa$ as follows. For $\eta \in \kappa^\kappa$ let $g(\eta)(\alpha) = 1$ if $h(\alpha) \in \eta$ and $g(\eta)(\alpha) = 0$ otherwise. Let us show that g is a continuous reduction $E_0^{<\lambda}(\kappa^\kappa) \leq_c E_0^{<\lambda}(2^\kappa)$. Suppose $\eta, \xi \in \kappa^\kappa$ are $E_0^{<\lambda}(\kappa^\kappa)$ -equivalent. Then clearly $|\eta \triangle \xi| < \lambda$. On the other hand

$$I = \{\alpha \mid g(\eta)(\alpha) \neq g(\xi)(\alpha)\} = \{\alpha \mid h(\alpha) \in \eta \triangle \xi\}$$

and because h is a bijection, we have that $|I| < \lambda$.

Suppose η and ξ are not $E_0^{<\lambda}(\kappa^\kappa)$ -equivalent. But then $|\eta \triangle \xi| \geq \lambda$ and the argument above shows that also $|I| \geq \lambda$, so $g(\eta)(\alpha)$ is not $E_0^{<\lambda}(2^\kappa)$ -equivalent to $g(\xi)(\alpha)$.

g is easily seen to be continuous. \square

We will need the following Lemma which is a straightforward generalization from the case $\kappa = \omega$:

33. Lemma. *Borel functions are continuous on a co-meager set.*

Proof. For each $\eta \in \kappa^{<\kappa}$ let V_η be an open subset of κ^κ such that $V_\eta \triangle f^{-1}N_\eta$ is meager. Let

$$D = \kappa^\kappa \setminus \bigcup_{\eta \in \kappa^{<\kappa}} V_\eta \triangle f^{-1}N_\eta.$$

Then D is as intended. Clearly it is co-meager, since we took away only a κ -union of meager sets. Let $\xi \in \kappa^{<\kappa}$ be arbitrary. The set $D \cap f^{-1}N_\xi$ is open in D since $D \cap f^{-1}N_\xi = D \cap V_\xi$ and so $f \upharpoonright D$ is continuous. \square

34. Theorem ($\kappa^{<\kappa} = \kappa$). $E_0^{<\lambda}$ is an equivalence relation on 2^κ for all $\lambda \leq \kappa$ and

- (1) $E_0^{<\lambda}$ is Borel.
- (2) $E_0^{<\kappa} \not\leq_B \text{id}$.
- (3) If $\lambda \leq \kappa$, then $\text{id} \leq_c E_0^{<\lambda}$.
- (4) If $\lambda < \kappa$, then $E_0^{<\lambda} \leq_c \text{id}$.

Proof. $E_0^{<\lambda}$ is clearly reflexive and symmetric. Suppose $\eta E_0^{<\lambda} \xi$ and $\xi E_0^{<\lambda} \zeta$. Denote $\eta = \eta^{-1}\{1\}$ and similarly for η, ζ . Then $|\eta \triangle \xi| < \lambda$ and $|\xi \triangle \zeta| < \lambda$; but $\eta \triangle \zeta \subset (\eta \triangle \xi) \cup (\xi \triangle \zeta)$. Thus $E_0^{<\lambda}$ is indeed an equivalence relation.

- (1) $E_0^{<\lambda} = \bigcup_{A \in [\kappa]^{<\lambda}} \bigcap_{\alpha \notin A} \underbrace{\{(\eta, \xi) \mid \eta(\alpha) = \xi(\alpha)\}}_{\text{open}}.$
- (2) Assume there were a Borel reduction $f: 2^\kappa \rightarrow 2^\kappa$ witnessing $E_0 \leq_B \text{id}$. By Lemma 33 there are dense open sets $(D_i)_{i < \kappa}$ such that $f \upharpoonright \bigcap_{i < \kappa} D_i$ is continuous. If $p, q \in 2^\alpha$ for some α and $\xi \in N_p$, let us denote $\xi^{(p/q)} = q \frown (\xi \upharpoonright (\kappa \setminus \alpha))$, and if $A \subset N_p$, denote

$$A^{(p/q)} = \{\eta^{(p/q)} \mid \eta \in A\}.$$

Let C be the collection of sets, each of which is of the form

$$\bigcup_{q \in 2^\alpha} [D_i \cap N_p]^{(p/q)}$$

for some $\alpha < \kappa$ and some $p \in 2^\alpha$. It is easy to see that each such set is dense and open, so C is a collection of dense open sets. By the assumption $\kappa^{<\kappa} = \kappa$, C has size κ . Also C contains the sets D_i for all $i < \kappa$, (taking $\alpha = 0$). Denote $D = \bigcap_{i < \kappa} D_i$. Let $\eta \in \bigcap C$, $\xi = f(\eta)$ and $\xi' \neq \xi$, $\xi' \in \text{ran}(f \upharpoonright D)$. Now ξ and ξ' have disjoint open neighbourhoods V and V' respectively. Let α and $p, q \in 2^\alpha$ be such that $\eta \in N_p$ and such that $D \cap N_p \subset f^{-1}[V]$ and $D \cap N_q \subset f^{-1}[V']$. These p and q exist by the continuity of f on D . Since $\eta \in \bigcap C$ and $\eta \in N_p$, we have

$$\eta \in [D_i \cap N_q]^{(q/p)}$$

for all $i < \kappa$, which is equivalent to

$$\eta^{(p/q)} \in [D_i \cap N_q]$$

for all $i < \kappa$, i.e. $\eta^{(p/q)}$ is in $D \cap N_q$. On the other hand (since $D_i \in C$ for all $i < \kappa$ and because $\eta \in N_p$), we have $\eta \in D \cap N_p$. This implies that $f(\eta) \in V$ and $f(\eta^{(p/q)}) \in V'$ which is a contradiction, because V and V' are disjoint and $(\eta, \eta^{(p/q)}) \in E_0$.

- (3) Let $(A_i)_{i < \kappa}$ be a partition of κ into pieces of size κ : if $i \neq j$ then $A_i \cap A_j = \emptyset$, $\bigcup_{i < \kappa} A_i = \kappa$ and $|A_i| = \kappa$. Obtain such a collection for instance by taking a bijection $h: \kappa \rightarrow \kappa \times \kappa$ and defining $A_i = h^{-1}[\kappa \times \{i\}]$. Let $f: 2^\kappa \rightarrow 2^\kappa$ be defined by $f(\eta)(\alpha) = \eta(i) \iff \alpha \in A_i$. Now if $\eta = \xi$, then clearly $f(\eta) = f(\xi)$ and so $f(\eta) E_0^{<\lambda} f(\xi)$. If $\eta \neq \xi$, then there exists i such that $\eta(i) \neq \xi(i)$ and we have that

$$A_i \subset \{\alpha \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\}$$

and A_i is of size $\kappa \geq \lambda$.

- (4) Let $P = \kappa^{<\kappa} \setminus \kappa^{<\lambda}$. Let $f: P \rightarrow \kappa$ be a bijection. It induces a bijection $g: 2^P \rightarrow 2^\kappa$. Let us construct a map $h: 2^\kappa \rightarrow 2^P$ such that $g \circ h$ is a reduction $E_0^{<\lambda} \rightarrow \text{id}_{2^\kappa}$. Let us denote by $E^{<\lambda}(\alpha)$ the equivalence relation on 2^α such that two subsets X, Y of α are $E^{<\lambda}(\alpha)$ -equivalent if and only if $|X \triangle Y| < \lambda$.

For each α in $\lambda < \alpha < \kappa$ let h_α be any reduction of $E^{<\lambda}(\alpha)$ to id_{2^α} . This exists because both equivalence relations have 2^α many classes. Now reduce $E_0^{<\lambda}$ to $\text{id}_{\kappa^{<\kappa}}$ by $f(A) = (h_\alpha(A \cap \alpha) \mid \lambda \leq \alpha < \kappa)$. If A, B are $E_0^{<\lambda}$ -equivalent, then $f(A) = f(B)$. Otherwise $f_\alpha(A \cap \alpha)$ differs from $f_\alpha(B \cap \alpha)$ for large enough $\alpha < \kappa$ because λ is less than κ and κ is regular. Continuity of h is easy to check. \square

4.2. On the Silver Dichotomy. To begin with, let us define the Silver Dichotomy and the Perfect Set Property:

35. Definition. Let $\mathcal{C} \in \{\text{Borel}, \Delta_1^1, \text{Borel}^*, \Sigma_1^1, \Pi_1^1\}$.

By the *Silver Dichotomy*, or more specifically, κ -SD for \mathcal{C} we mean the statement that there are no equivalence relations E in the class \mathcal{C} such that $E \subset 2^\kappa \times 2^\kappa$ and E has more than κ equivalence classes such that $\text{id} \not\leq_B E$, $\text{id} = \text{id}_{2^\kappa}$.

Similarly the *Perfect Set Property*, or κ -PSP for \mathcal{C} , means that each member A of \mathcal{C} has either size $\leq \kappa$ or there is a Borel injection $2^\kappa \rightarrow A$. Using Lemma 33 it is not hard to see that this definition is equivalent to the game definition given in [22].

4.2.1. The Silver Dichotomy for Isomorphism Relations. Although the Silver Dichotomy for Borel sets is not provable from ZFC for $\kappa > \omega$ (see Theorem 42 on page 35), it holds when the equivalence relation is an isomorphism relation, if $\kappa > \omega$ is an inaccessible cardinal:

36. Theorem. Assume that κ is inaccessible. If the number of equivalence classes of \cong_T is greater than κ , then $\text{id} \leq_c \cong_T$.

Proof. Suppose that there are more than κ equivalence classes of \cong_T . We will show that then $\text{id}_{2^\kappa} \leq_c \cong_T$. If T is not classifiable, then as was done in [26], we can construct a tree $t(S)$ for each $S \subset S_\omega^\kappa$ and Ehrenfeucht-Mostowski-type models $M(t(S))$ over these trees such that if $S \triangle S'$ is stationary, then $M(t(S)) \not\cong M(t(S'))$. Now it is easy to construct a reduction $f: \text{id}_{2^\kappa} \leq_c E_{S_\omega^\kappa}$ (see notation defined in Section 2.1), so then $\eta \mapsto M(t(f(\eta)))$ is a reduction $\text{id} \leq_c \cong_T$.

Assume now that T is classifiable. By $\lambda(T)$ we denote the least cardinal in which T is stable. By [25] Theorem XIII.4.8 (this is also mentioned in [7] Theorem 2.5), assuming that \cong_T has more than κ equivalence classes, it has depth at least 2 and so there are: a $\lambda(T)^+$ -saturated model $\mathcal{B} \models T$, $|\mathcal{B}| = \lambda(T)$, and a $\lambda(T)^+$ -saturated elementary submodel $\mathcal{A} \preceq \mathcal{B}$ and $a \notin \mathcal{B}$ such that $\text{tp}(a/\mathcal{B})$ is orthogonal to \mathcal{A} . Let $f: \kappa \rightarrow \kappa$ be strictly increasing and such that for all $\alpha < \kappa$, $f(\alpha) = \mu^+$, for some μ with the properties $\lambda(T) < \mu < \kappa$, $\text{cf}(\mu) = \mu$ and $\mu^{2^\omega} = \mu$. For each $\eta \in 2^\kappa$ with $\eta^{-1}\{1\}$ is unbounded we will construct a model \mathcal{A}_η . As above, it will be enough to show that $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$ whenever $\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is λ -stationary where $\lambda = \lambda(T)^+$. Fix $\eta \in 2^\kappa$ and let $\lambda = \lambda(T)^+$.

For each $\alpha \in \eta^{-1}\{1\}$ choose $\mathcal{B}_\alpha \supset \mathcal{A}$ such that

- (1) $\exists \pi_\alpha: \mathcal{B} \cong \mathcal{B}_\alpha$, $\pi_\alpha \upharpoonright \mathcal{A} = \text{id}_\mathcal{A}$.
- (2) $\mathcal{B}_\alpha \downarrow_\mathcal{A} \bigcup \{\mathcal{B}_\beta \mid \beta \in \eta^{-1}\{1\}, \beta \neq \alpha\}$

Note that 2 implies that if $\alpha \neq \beta$, then $\mathcal{B}_\alpha \cap \mathcal{B}_\beta = \mathcal{A}$. For each $\alpha \in \eta^{-1}\{1\}$ and $i < f(\alpha)$ choose tuples a_i^α with the properties

- (3) $\text{tp}(a_i^\alpha/\mathcal{B}_\alpha) = \pi_\alpha(\text{tp}(a/\mathcal{B}))$
- (4) $a_i^\alpha \downarrow_{\mathcal{B}_\alpha} \bigcup \{a_j^\alpha \mid j < f(\alpha), j \neq i\}$

Let \mathcal{A}_η be F_λ^s -primary over

$$S_\eta = \bigcup \{B_\alpha \mid a < \eta^{-1}\{1\}\} \cup \bigcup \{a_i^\alpha \mid \alpha < \eta^{-1}\{1\}, i < f(\alpha)\}.$$

It remains to show that if $S_\lambda^\kappa \cap \eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is stationary, then $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$. Without loss of generality we may assume that $S_\lambda^\kappa \cap \eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$ is stationary. Let us make a counter assumption, namely that there is an isomorphism $F: \mathcal{A}_\eta \rightarrow \mathcal{A}_\xi$.

Without loss of generality there exist singletons b_i^η and sets B_i^η , $i < \kappa$ of size $< \lambda$ such that $\mathcal{A}_\eta = S_\eta \cup \bigcup_{i < \kappa} b_i^\eta$ and $(S_\eta, (b_i^\eta, B_i^\eta)_{i < \kappa})$ is an F_λ^s -construction.

Let us find an ordinal $\alpha < \kappa$ and sets $C \subset \mathcal{A}_\eta$ and $D \subset \mathcal{A}_\xi$ with the properties listed below:

- (a) $\alpha \in \eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$
- (b) $D = F[C]$
- (c) $\forall \beta \in (\alpha + 1) \cap \eta^{-1}\{1\} (\mathcal{B}_\beta \subset C)$ and $\forall \beta \in (\alpha + 1) \cap \xi^{-1}\{1\} (\mathcal{B}_\beta \subset D)$,
- (d) for all $i < f(\alpha)$, $\forall \beta \in \alpha \cap \eta^{-1}\{1\} (a_i^\beta \in C)$ and $\forall \beta \in \alpha \cap \xi^{-1}\{1\} (a_i^\beta \in D)$,
- (e) $|C| = |D| < f(\alpha)$,
- (f) For all β , if $\mathcal{B}_\beta \cap C \setminus \mathcal{A} \neq \emptyset$, then $\mathcal{B}_\beta \subset C$ and if $\mathcal{B}_\beta \cap D \setminus \mathcal{A} \neq \emptyset$, then $\mathcal{B}_\beta \subset D$,

- (g) C and D are λ -saturated,
- (h) if $b_i^\eta \in C$, then $B_i^\eta \subset [S_\eta \cup \bigcup\{b_i^\eta \mid j < i\}] \cap C$ and if $b_i^\xi \in D$, then $B_i^\xi \subset [S_\xi \cup \bigcup\{b_i^\xi \mid j < i\}] \cap D$.

This is possible, because $\eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$ is stationary and we can close under the properties (b)–(h).

Now \mathcal{A}_η is F_λ^s -primary over $C \cup S_\eta$ and \mathcal{A}_ξ is F_λ^s -primary over $D \cup S_\eta$ and thus \mathcal{A}_η is F_λ^s -atomic over $C \cup S_\eta$ and \mathcal{A}_ξ is F_λ^s -atomic over $D \cup S_\xi$. Let

$$I_\alpha = \{a_i^\alpha \mid i < f(\alpha)\}.$$

Now $|I_\alpha \setminus C| = f(\alpha)$, because $|C| < f(\alpha)$, and so $I_\alpha \setminus C \neq \emptyset$. Let $c \in I_\alpha \setminus C$ and let $A \subset S_\xi \setminus D$ and $B \subset D$ be such that $\text{tp}(F(c)/A \cup B) \vdash \text{tp}(F(c)/D \cup S_\xi)$ and $|A \cup B| < \lambda$. Since $\alpha \notin \xi^{-1}\{1\}$, we can find (just take disjoint copies) a sequence $(A_i)_{i < f(\alpha)^+}$ such that $A_i \subset I_\alpha \cap \mathcal{A}_\xi$, $\text{tp}(A_i/D) = \text{tp}(A/D)$ and $A_i \downarrow_D \bigcup\{A_j \mid j \neq i, j < f(\alpha)^+\}$.

Now we can find $(d_i)_{i < f(\alpha)^+}$, such that

$$\text{tp}(d_i \frown A_i \frown B_i / \emptyset) = \text{tp}(F(c) \frown A \frown B / \emptyset).$$

Then it is a Morley sequence over D and for all $i < f(\alpha)^+$,

$$\text{tp}(d_i/D) = \text{tp}(F(c)/D),$$

which implies

$$\text{tp}(F^{-1}(d_i)/C) = \text{tp}(c/C),$$

for some i , since for some i we have $c = a_i^\alpha$. Since by (c), $\mathcal{B}_\alpha \subset C$, the above implies that

$$\text{tp}(F^{-1}(d_i)/\mathcal{B}_\alpha) = \text{tp}(a_i^\alpha/\mathcal{B}_\alpha)$$

which by the definition of a_i^α , item 3 implies

$$\text{tp}(F^{-1}(d_i)/\mathcal{B}_\alpha) = \pi_\alpha(\text{tp}(a/\mathcal{B})).$$

Thus the sequence $(F^{-1}(d_i))_{i < f(\alpha)^+}$ witnesses that the dimension of $\pi_\alpha(\text{tp}(a/\mathcal{B}))$ in \mathcal{A}_η is greater than $f(\alpha)$. Denote that sequence by J . Since $\pi_\alpha(\text{tp}(a/\mathcal{B}))$ is orthogonal to \mathcal{A} , we can find $J' \subset J$ such that $|J'| = f(\alpha)^+$ and J' is a Morley sequence over S_η . Since $f(\alpha)^+ > \lambda$, this contradicts Theorem 4.9(2) of Chapter IV of [25]. \square

Open Problem. Under what conditions on κ does the conclusion of Theorem 36 hold?

4.2.2. *Theories Bireducible With id.*

37. Theorem. *Assume $\kappa^{<\kappa} = \kappa = \aleph_\alpha > \omega$, κ is not weakly inaccessible and $\lambda = |\alpha + \omega|$. Then the following are equivalent*

- (1) *There is $\gamma < \omega_1$ such that $\beth_\gamma(\lambda) \geq \kappa$.*
- (2) *There is a complete countable T such that $\text{id} \leq_B \cong_T$ and $\cong_T \leq_B \text{id}$.*

Proof. (2) \Rightarrow (1): Suppose that (1) is not true. Notice that then $\kappa > 2^\omega$. Then every shallow classifiable theory has $< \kappa$ many models of power κ (see [7], item 6. of the Theorem which is on the first page of the article.) and thus $\text{id} \not\leq_B \cong_T$. On the other hand if T is not classifiable and shallow, \cong_T is not Borel by Theorem 70 and thus it is not Borel reducible to id by Fact 6.

(1) \Rightarrow (2): Since $\text{cf}(\kappa) > \omega$, (1) implies that there is $\alpha = \beta + 1 < \omega_1$ such that $\beth_\alpha(\lambda) = \kappa$. But then there is an L^* -theory T^* which has exactly κ many models in cardinality κ (up to isomorphism, use [7], Theorem 6.1 items 2. and 8.). But then it has exactly κ many models of cardinality $\leq \kappa$, let \mathcal{A}_i , $i < \kappa$, list these. Such a theory must be classifiable and shallow. Let L be the vocabulary we get from L^* by adding one binary relation symbol E . Let \mathcal{A} be an L -structure in which E is an equivalence relation with infinitely many equivalence classes such that for every equivalence class a/E , $(\mathcal{A} \upharpoonright a/E) \upharpoonright L^*$ is a model of T^* . Let $T = Th(\mathcal{A})$.

We show first that identity on $\{\eta \in 2^\kappa \mid \eta(0) = 1\}$ reduces to \cong_T . For all $\eta \in 2^\kappa$, let \mathcal{B}_η be a model of T of power κ such that if $\eta(i) = 0$, then the number of equivalence classes isomorphic to \mathcal{B}_i is countable and otherwise the number is κ . Clearly we can code \mathcal{B}_η as $\xi_\eta \in 2^\kappa$ so that $\eta \mapsto \xi_\eta$ is the required Borel reduction.

We show then that \cong_T Borel reduces to identity on

$$X = \{\eta: \kappa \rightarrow (\kappa + 1)\}.$$

Since T^* is classifiable and shallow, for all $\delta, i < \kappa$ the set

$$\{\eta \in X \mid (\mathcal{A}_\eta \upharpoonright \delta/E) \upharpoonright L^* \cong \mathcal{A}_i\}$$

is Borel. But then for all cardinals $\theta \leq \kappa$ and $i < \kappa$, the set

$$\{\eta \in X \mid \text{card}(\{\delta/E \mid \delta < \kappa, (\mathcal{A}_\eta \upharpoonright \delta/E) \upharpoonright L^* \cong \mathcal{A}_i\}) = \theta\}$$

is Borel. But then $\eta \mapsto \xi_\eta$ is the required reduction when

$$\xi_\eta(i) = |\{\delta/E \mid \delta < \kappa, (\mathcal{A}_\eta \upharpoonright \delta/E) \upharpoonright L^* \cong \mathcal{A}_i\}|.$$

□

4.2.3. *Failures of Silver's Dichotomy.* There are well-known dichotomy theorems for Borel equivalence relations on 2^ω . Two of them are:

38. **Theorem** (Silver, [31]). *Let $E \subset 2^\omega \times 2^\omega$ be a Π_1^1 equivalence relation. If E has uncountably many equivalence classes, then $\text{id}_{2^\omega} \leq_B E$.* \square

39. **Theorem** (Generalized Glimm-Effros dichotomy, [6]). *Let $E \subset 2^\omega \times 2^\omega$ be a Borel equivalence relation. Then either $E \leq_B \text{id}_{2^\omega}$ or else $E_0 \leq_c E$.* \square

As in the case $\kappa = \omega$ we have the following also for uncountable κ (see Definition 35):

40. **Theorem.** *If κ -SD for Π_1^1 holds, then the κ -PSP holds for Σ_1^1 -sets. More generally, if $\mathcal{C} \in \{\text{Borel}, \Delta_1^1, \text{Borel}^*, \Sigma_1^1, \Pi_1^1\}$, then κ -SD for \mathcal{C} implies κ -PSP for \mathcal{C}' , where elements in \mathcal{C}' are all the complements of those in \mathcal{C} .*

Proof. Let us prove this for $\mathcal{C} = \Pi_1^1$, the other cases are similar. Suppose we have a Σ_1^1 -set A . Let

$$E = \{(\eta, \xi) \mid \eta = \xi \text{ or } ((\eta \notin A) \wedge (\xi \notin A))\}.$$

Now $E = \text{id} \cup (2^\kappa \setminus A)^2$. Since A is Σ_1^1 , $(2^\kappa \setminus A)^2$ is Π_1^1 and because id is Borel, also E is Π_1^1 . Obviously $|A|$ is the number of equivalence classes of E provided A is infinite. Then suppose $|A| > \kappa$. Then there are more than κ equivalence classes of E , so by κ -SD for Π_1^1 , there is a reduction $f: \text{id} \leq E$. This reduction in fact witnesses the PSP of A . \square

The idea of using Kurepa trees for this purpose arose already in the paper [22] by Mekler and Väänänen.

41. **Definition.** If $t \subset 2^{<\kappa}$ is a tree, a *path* through t is a branch of length κ . A κ -Kurepa tree is a tree $K \subset 2^{<\kappa}$ which satisfies the following:

- (a) K has more than κ paths,
- (b) K is downward closed,
- (c) for all $\alpha < \kappa$, the levels are small: $|\{p \in K \mid \text{dom } p = \alpha\}| \leq |\alpha + \omega|$.

42. **Theorem.** *Assume one of the following:*

- (1) κ is regular but not strongly inaccessible and there exists a κ -Kurepa tree $K \subset 2^{<\kappa}$,
- (2) κ is regular (might be strongly inaccessible), $2^\kappa > \kappa^+$ and there exists a tree $K \subset 2^{<\kappa}$ with more than κ but less than 2^κ branches.

Then the Silver Dichotomy for κ does not hold. In fact there an equivalence relation $E \subset 2^\kappa \times 2^\kappa$ which is the union of a closed and an open set, has more than κ equivalence classes but $\text{id}_{2^\kappa} \not\leq_B E$.

Proof. Let us break the proof according to the assumptions (1) and (2). So first let us consider the case where κ is not strongly inaccessible and there is a κ -Kurepa tree.

(1): Let us carry out the proof in the case $\kappa = \omega_1$. It should be obvious then how to generalize it to any κ not strongly inaccessible. So let $K \subset 2^{<\omega_1}$ be an ω_1 -Kurepa tree. Let P be the collection of all paths of K . For $b \in P$, denote $b = \{b_\alpha \mid \alpha < \omega_1\}$ where b_α is an element of K with domain α .

Let

$$C = \{\eta \in 2^{\omega_1} \mid \eta = \bigcup_{\alpha < \omega_1} b_\alpha, b \in P\}.$$

Clearly C is closed.

Let $E = \{(\eta, \xi) \mid (\eta \notin C \wedge \xi \notin C) \vee (\eta \in C \wedge \eta = \xi)\}$. In words, E is the equivalence relation whose equivalence classes are the complement of C and the singletons formed by the elements of C . E is the union of the open set $\{(\eta, \xi) \mid \eta \notin C \wedge \xi \notin C\}$ and the closed set $\{(\eta, \xi) \mid \eta \in C \wedge \eta = \xi\} = \{(\eta, \eta) \mid \eta \in C\}$. The number of equivalence classes equals the number of paths of K , so there are more than ω_1 of them by the definition of Kurepa tree.

Let us show that $\text{id}_{2^{\omega_1}}$ is not embeddable to E . Suppose that $f: 2^{\omega_1} \rightarrow 2^{\omega_1}$ is a Borel reduction. We will show that then K must have a level of size $\geq \omega_1$ which contradicts the definition of Kurepa tree. By Lemma 33 there is a co-meager set D on which $f \upharpoonright D$ is continuous. There is at most one $\eta \in 2^{\omega_1}$ whose image $f(\eta)$ is outside C , so without loss of generality $f[D] \subset C$. Let p be an arbitrary element of K such that $f^{-1}[N_p] \neq \emptyset$. By continuity there is a $q \in 2^{<\omega_1}$ with $f[N_q \cap D] \subset N_p$. Since D is co-meager, there are η and ξ such that $\eta \neq \xi$, $q \subset \eta$ and $q \subset \xi$. Let $\alpha_1 < \omega_1$ and p_0 and p_1 be extensions of p with the properties $p_0 \subset f(\eta)$, $p_1 \subset f(\xi)$, $\alpha_1 = \text{dom } p_0 = \text{dom } p_1$, $f^{-1}[N_{p_0}] \neq \emptyset \neq f^{-1}[N_{p_1}]$ and $N_{p_0} \cap N_{p_1} = \emptyset$. Note that p_0 and p_1 are in K . Then, again by continuity, there are q_0 and q_1 such that $f[N_{q_0} \cap D] \subset N_{p_0}$ and $f[N_{q_1} \cap D] \subset N_{p_1}$. Continue in the same manner to obtain α_n and $p_s \in K$ for each $n < \omega$ and $s \in 2^{<\omega}$ so that $s \subset s' \iff p_s \subset p_{s'}$ and $\alpha_n = \text{dom } p_s \iff n = \text{dom } s$. Let $\alpha = \sup_{n < \omega} \alpha_n$. Now clearly the α 's level of K contains continuum many elements: by (b) in the

definition of Kurepa tree it contains all the elements of the form $\bigcup_{n < \omega} p_{\eta \upharpoonright n}$ for $\eta \in 2^\omega$ and $2^\omega \geq \omega_1$.

If κ is arbitrary regular not strongly inaccessible cardinal, then the proof is the same, only instead of ω steps one has to do λ steps where λ is the least cardinal satisfying $2^\lambda \geq \kappa$.

(2): The argument is even simpler. Define the equivalence relation E exactly as above. Now E is again closed and has as many equivalence classes as is the number of paths in K . Thus the number of equivalence classes is $> \kappa$ but it cannot be reduced to E since there are less than 2^κ equivalence classes. \square

Remark. Some related results:

- (1) In L , the PSP fails for closed sets for all uncountable regular κ . This is because “weak Kurepa trees” exist (see the proof sketch of (3) below for the definition of “weak Kurepa tree”).
- (2) (P. Schlicht) In Silver’s model where an inaccessible κ is made into ω_2 by Levy collapsing each ordinal below to ω_1 with countable conditions, every Σ_1^1 subset X of 2^{ω_1} obeys the PSP.
- (3) Supercompactness does not imply the PSP for closed sets.

Sketch of a proof of item (3). Suppose κ is supercompact and by a reverse Easton iteration add to each inaccessible α a “weak Kurepa tree”, i.e., a tree T_α with α^+ branches whose β^{th} level has size β for stationary many $\beta < \alpha$. The forcing at stage α is α -closed and the set of branches through T_κ is a closed set with no perfect subset. If $j: V \rightarrow M$ witnesses λ -supercompactness ($\lambda > \kappa$) and G is the generic then we can find G^* which is $j(P)$ -generic over M containing $j[G]$: Up to λ we copy G , between λ and $j(\kappa)$ we build G^* using λ^+ closure of the forcing and of the model M , and at $j(\kappa)$ we form a master condition out of $j[G(\kappa)]$ and build a generic below it, again using λ^+ closure. \square

43. Corollary. *The consistency of the Silver Dichotomy for Borel sets on ω_1 with CH implies the consistency of a strongly inaccessible cardinal. In fact, if there is no equivalence relation witnessing the failure of the Silver Dichotomy for ω_1 , then ω_2 is inaccessible in L .*

Proof. By a result of Silver, if there are no ω_1 -Kurepa trees, then ω_2 is inaccessible in L , see Exercise 27.5 in Part III of [15]. \square

Open Problem. Is the Silver Dichotomy for uncountable κ consistent?

4.3. Regularity Properties and Definability of the CUB Filter. In the standard descriptive theory ($\kappa = \omega$), the notions of Borel, Δ_1^1 and Borel* coincide and one of the most important observations in the theory is that such sets have the Property of Baire and that the Σ_1^1 -sets obey the Perfect Set Property. In the case $\kappa > \omega$ the situation is more complicated as the following shows. It was already pointed out in the previous section that $\text{Borel} \subsetneq \Delta_1^1$. In this section we focus on the cub filter

$$\text{CUB} = \{\eta \in 2^\kappa \mid \eta^{-1}\{1\} \text{ contains a cub}\}.$$

The set CUB is easily seen to be Σ_1^1 : the set

$$\{(\eta, \xi) \mid (\eta^{-1}\{1\} \subset \xi^{-1}\{1\}) \wedge (\eta^{-1}\{1\} \text{ is cub})\}$$

is Borel. CUB (restricted to cofinality ω , see Definition 48) will serve (consistently) as a counterexample to $\Delta_1^1 = \text{Borel}^*$, but we will show that it is also consistent that CUB is Δ_1^1 . The latter implies that it is consistent that Δ_1^1 -sets do not have the Property of Baire and we will also show that in a forcing extension of L , Δ_1^1 -sets all have the Property of Baire.

44. Definition. A *nowhere dense set* is a subset of a set whose complement is dense and open. Let $X \subset \kappa^\kappa$. A subset $M \subset X$ is κ -*meager in X*, if $M \cap X$ is the union of no more than κ nowhere dense sets,

$$M = \bigcup_{i < \kappa} N_i.$$

We usually drop the prefix “ κ -”.

Clearly κ -meager sets form a κ -complete ideal. A *co-meager* set is a set whose complement is meager.

A subset $A \subset X$ has the *Property of Baire* or shorter *P.B.*, if there exists an open $U \subset X$ such that the symmetric difference $U \triangle A$ is meager.

Halko showed in [4] that

45. Theorem ([4]). *Borel sets have the Property of Baire.* □

(The same proof as when $\kappa = \omega$ works.) This is independent of the assumption $\kappa^{<\kappa} = \kappa$. Borel* sets do not in general have the Property of Baire.

46. Definition ([21, 22, 10]). A $\kappa^+\kappa$ -tree t is a $\kappa\lambda$ -*canary tree* if for all stationary $S \subset S_\lambda^\kappa$ it holds that if \mathbb{P} does not add subsets of κ of size less than κ and \mathbb{P} kills the stationarity of S , then \mathbb{P} adds a κ -branch to t .

Remark. Hyttinen and Rautila [10] use the notation κ -canary tree for our $\kappa^+\kappa$ -canary tree.

It was shown by Mekler and Shelah [21] and Hyttinen and Rautila [10] that it is consistent with ZFC+GCH that there is a $\kappa^+\kappa$ -canary tree *and* it is consistent with ZFC+GCH that there are no $\kappa^+\kappa$ -canary trees. The same proof as in [21, 10] gives the following:

47. Theorem. *Assume GCH and assume $\lambda < \kappa$ are regular cardinals. Let \mathbb{P} be the forcing which adds κ^+ Cohen subsets of κ . Then in the forcing extension there are no $\kappa\lambda$ -canary trees.* \square

48. Definition. Suppose $X \subset \kappa$ is stationary. For each such X define the set

$$\text{CUB}(X) = \{\eta \in 2^\kappa \mid X \setminus \eta^{-1}\{1\} \text{ is non-stationary}\},$$

so $\text{CUB}(X)$ is “cub in X ”.

49. Theorem. *In the following κ satisfies $\kappa^{<\kappa} = \kappa > \omega$.*

- (1) $\text{CUB}(S_\omega^\kappa)$ is Borel*.
- (2) For all regular $\lambda < \kappa$, $\text{CUB}(S_\lambda^\kappa)$ is not Δ_1^1 in the forcing extension after adding κ^+ Cohen subsets of κ .
- (3) If $V = L$, then for every stationary $S \subset \kappa$, the set $\text{CUB}(S)$ is not Δ_1^1 .
- (4) Assume GCH and that κ is not a successor of a singular cardinal. For any stationary set $Z \subset \kappa$ there exists a forcing notion \mathbb{P} which has the κ^+ -c.c., does not add bounded subsets of κ and preserves GCH and stationary subsets of $\kappa \setminus Z$ such that $\text{CUB}(\kappa \setminus Z)$ is Δ_1^1 in the forcing extension.
- (5) Let the assumptions for κ be as in (4). For all regular $\lambda < \kappa$, $\text{CUB}(S_\lambda^\kappa)$ is Δ_1^1 in a forcing extension as in (4).
- (6) $\text{CUB}(X)$ does not have the Property of Baire for stationary $X \subset \kappa$. (Proved by Halko and Shelah in [5] for $X = \kappa$)
- (7) It is consistent that all Δ_1^1 -sets have the Property of Baire. (Independently known to P. Lücke and P. Schlicht.)

Proof of Theorem 49.

Proof of item (1). Let $t = [\kappa]^{<\omega}$ (increasing functions ordered by end extension) and for all branches $b \subset t$

$$h(b) = \{\xi \in 2^\kappa \mid \xi(\sup_{n < \omega} b(n)) \neq 0\}.$$

Now if $\kappa \setminus \xi^{-1}\{0\}$ contains an ω -cub set C , then player **II** has a winning strategy in $G(t, h, \xi)$: for her n^{th} move she picks an element $x \in t$ with domain $2n + 2$ such that $x(2n + 1)$ is in C . Suppose the players picked a branch b in this way. Then the condition $\xi(b(2n + 1)) \neq 0$ holds for all $n < \omega$ and because C is cub outside $\xi^{-1}\{0\}$, we have $\xi(\sup_{n < \omega} b(n)) \neq 0$.

Suppose on the contrary that $S = \xi^{-1}\{0\}$ is stationary. Let σ be any strategy of player **II**. Let C_σ be the set of ordinals closed under this strategy. It is a cub set, so there is an $\alpha \in C_\sigma \cap S$. Player **I** can now easily play towards this ordinal to force $\xi(b(\omega)) = 0$, so σ cannot be a winning strategy. $\square_{\text{item (1)}}$

Proof of item (2). It is not hard to see that $\text{CUB}_\lambda^\kappa$ is Δ_1^1 if and only if there exists a $\kappa\lambda$ -canary tree. This fact is proved in detail in [22] in the case $\kappa = \omega_1$, $\lambda = \omega$ and the proof generalizes easily to any regular uncountable κ along with the assumption $\kappa^{<\kappa} = \kappa$. So the statement follows from Theorem 47. $\square_{\text{item (2)}}$

Proof of item (3). Suppose that φ is Σ_1 and for simplicity assume that φ has no parameters. Then for $x \subset \kappa$ we have:

Claim. $\varphi(x)$ holds if and only if the set A of those α for which there exists $\beta > \alpha$ such that

$$L_\beta \models (ZF^- \wedge (\omega < \alpha \text{ is regular}) \wedge ((S \cap \alpha) \text{ is stationary}) \wedge \varphi(x \cap \alpha))$$

contains $C \cap S$ for some cub set C .

Proof of the Claim. “ \implies ”. If $\varphi(x)$ holds then choose a continuous chain $(M_i \mid i < \kappa)$ of elementary submodels of some large ZF^- model L_θ so that x and S belong to M_0 and the intersection of each M_i with κ is an ordinal α_i less than κ . Let C be the set of α_i ’s, cub in κ . Then any α in $C \cap S$ belongs to A by condensation.

“ \impliedby ”. If $\varphi(x)$ fails then let C be any cub in κ and let D be the cub of $\alpha < \kappa$ such that $H(\alpha)$ is the Skolem Hull in some large L_θ of α together with $\{\kappa, S, C\}$ contains no ordinals in the interval $[\alpha, \kappa)$. Let α be the least element of $S \cap \lim(D)$. Then α does not belong to A : If L_β satisfies $\varphi(x \cap \alpha)$ then β must be greater than $\bar{\beta}$ where $\overline{H(\alpha)} = L_{\bar{\beta}}$ is the transitive collapse of $H(\alpha)$, because $\varphi(x \cap \alpha)$ fails in $\overline{H(\alpha)}$. But as $\lim(D) \cap \alpha$ is an element of $L_{\bar{\beta}+2}$ and is disjoint from S , it follows that either α is singular in L_β or $S \cap \alpha$ is not stationary in $L_{\bar{\beta}+2}$ and hence not in L_β . Of course α does belong to C so we have shown that A does not contain $S \cap C$ for an arbitrary cub C in κ . \square_{Claim}

It follows from the above that any Σ_1 subset of 2^κ is Δ_1 over $(L_\kappa^+, \text{CUB}(S))$ and therefore if $\text{CUB}(S)$ were Δ_1 then any Σ_1 subset of 2^κ would be Δ_1 , a contradiction. $\square_{\text{item (3)}}$

Proof of item (4). If $X \subset 2^\kappa$ is Δ_1^1 , then $\{\eta \in X \mid \eta^{-1}\{1\} \subset \kappa \setminus Z\}$ is Δ_1^1 , so it is sufficient to show that we can force a set $E \subset Z$ which has the claimed property. So we force a set $E \subset Z$ such that E is stationary but $E \cap \alpha$ is non-stationary in α for all $\alpha < \kappa$ and $\kappa \setminus E$ is fat. A set is *fat* if its intersection with any cub set contains closed increasing sequences of all order types $< \kappa$.

This can be easily forced with

$$\mathbb{R} = \{p: \alpha \rightarrow 2 \mid \alpha < \kappa, p^{-1}\{1\} \cap \beta \subset Z \text{ is non-stationary in } \beta \text{ for all } \beta \leq \alpha\}$$

ordered by end-extension. It is easy to see that for any \mathbb{R} -generic G the set $E = (\cup G)^{-1}\{1\}$ satisfies the requirements. Also \mathbb{R} does not add bounded subsets of κ and has the κ^+ -c.c. and does not kill stationary sets.

Without loss of generality assume that such E exists in V and that $0 \in E$.

Next let $\mathbb{P}_0 = \{p: \alpha \rightarrow 2^{<\alpha} \mid \alpha < \kappa, p(\beta) \in 2^\beta, p(\beta)^{-1}\{1\} \subset E\}$. This forcing adds a \diamond_E -sequence $\langle A_\alpha \mid \alpha \in E \rangle$ (if G is generic, set $A_\alpha = (\cup G)(\alpha)^{-1}\{1\}$) such that for all $B \subset E$ there is a stationary $S \subset E$ such that $A_\alpha = B \cap \alpha$ for all $\alpha \in S$. This forcing \mathbb{P}_0 is $< \kappa$ -closed and clearly has the κ^+ -c.c., so it is easily seen that it does not add bounded subsets of κ and does not kill stationary sets.

Let $\psi(G, \eta, S)$ be a formula with parameters $G \in (2^{<\kappa})^\kappa$ and $\eta \in 2^\kappa$ and a free variable $S \subset \kappa$ which says:

$$\forall \alpha < \kappa (\alpha \in S \iff G(\alpha)^{-1}\{1\} = \eta^{-1}\{1\} \cap \alpha).$$

If $\langle G(\alpha)^{-1}\{1\} \rangle_{\alpha < \kappa}$ happens to be a \diamond_E -sequence, then S satisfying ψ is always stationary. Thus if G_0 is \mathbb{P}_0 -generic over V and $\eta \in 2^E$, then $(\psi(G_0, \eta, S) \rightarrow (S \text{ is stationary}))^{V[G_0]}$.

For each $\eta \in 2^E$, let \dot{S}_η be a nice \mathbb{P}_0 -name for the set S such that $V[G_0] \models \psi(G_0, \eta, S)$ where G_0 is \mathbb{P}_0 -generic over V . By the definitions, $\mathbb{P}_0 \Vdash \text{“}\dot{S}_\eta \subset \check{E} \text{ is stationary”}$ and if $\eta \neq \eta'$, then $\mathbb{P}_0 \Vdash \text{“}\dot{S}_\eta \cap \dot{S}_{\eta'} \text{ is bounded”}$.

Let us enumerate $E = \{\beta_i \mid i < \kappa\}$ such that $i < j \Rightarrow \beta_i < \beta_j$ and for $\eta \in 2^E$ and $\gamma \in \kappa$ define $\eta + \gamma$ to be the $\xi \in 2^E$ such that $\xi(\beta_i) = 1$ for all $i < \gamma$ and $\xi(\beta_{\gamma+j}) = \eta(\beta_j)$ for $j \geq 0$. Let

$$F_0 = \{\eta \in 2^E \mid \eta(0) = 0\}^V. \quad (*)$$

Now for all $\eta, \eta' \in F_0$ and $\alpha, \alpha' \in \kappa$, $\eta + \alpha = \eta' + \alpha'$ implies $\eta = \eta'$ and $\alpha = \alpha'$. Let us now define the formula $\varphi(G, \eta, X)$ with parameters $G \in (2^{<\kappa})^\kappa$, $\eta \in 2^\kappa$ and a free variable $X \subset \kappa \setminus E$ which says:

$$(\eta(0) = 0) \wedge \forall \alpha < \kappa [(\alpha \in X \rightarrow \exists S(\psi(G, \eta + 2\alpha, S) \wedge S \text{ is non-stationary})) \\ \wedge (\alpha \notin X \rightarrow \exists S(\psi(G, \eta + 2\alpha + 1, S) \wedge S \text{ is non-stationary}))].$$

Now, we will construct an iterated forcing \mathbb{P}_{κ^+} , starting with \mathbb{P}_0 , which kills the stationarity of \dot{S}_η for suitable $\eta \in 2^E$, such that if G is \mathbb{P}_{κ^+} -generic, then for all $S \subset \kappa \setminus E$, S is stationary if and only if

$$\exists \eta \in 2^E (\varphi(G_0, \eta, S))$$

where $G_0 = G \upharpoonright \{0\}$. In this model, for each $\eta \in F_0$, there will be a unique X such that $\varphi(G_0, \eta, X)$, so let us denote this X by X_η . It is easy to check that the mapping $\eta \mapsto X_\eta$ defined by φ is Σ_1^1 so in the result, also $\mathcal{S} = \{S \subset \kappa \setminus E \mid S \text{ is stationary}\}$ is Σ_1^1 . Since cub and non-stationarity are also Σ_1^1 , we get that \mathcal{S} is Δ_1^1 , as needed.

Let us show how to construct the iterated forcing. For $S \subset \kappa$, we denote by $T(S)$ the partial order of all closed increasing sequences contained in the complement of S . Clearly $T(S)$ is a forcing that kills the stationarity of S . If the complement of S is fat and S is non-reflecting, then $T(S)$ has all the nice properties we need, as the following claims show. Let $f: \kappa^+ \setminus \{0\} \rightarrow \kappa^+ \times \kappa^+$ be a bijection such that $f_1(\gamma) \leq \gamma$.

\mathbb{P}_0 is already defined and it has the κ^+ -c.c. and it is $< \kappa$ -closed. Suppose that \mathbb{P}_i has been defined for $i < \alpha$ and σ_i has been defined for $i < \cup \alpha$ such that σ_i is a (nice) \mathbb{P}_i -name for a κ^+ -c.c. partial order. Also suppose that for all $i < \cup \alpha$, $\{(\dot{S}_{ij}, \delta_{ij}) \mid j < \kappa^+\}$ is the list of all pairs (\dot{S}, δ) such that \dot{S} is a nice \mathbb{P}_i -name for a subset of $\kappa \setminus \check{E}$ and $\delta < \kappa$, and suppose that

$$g_\alpha: \{\dot{S}_{f(i)} \mid i < \alpha\} \rightarrow F_0 \tag{***}$$

is an injective function, where F_0 is defined at (*).

If α is a limit, let \mathbb{P}_α consist of those $p: \alpha \rightarrow \bigcup_{i < \alpha} \text{dom } \sigma_i$ with $|\text{sprt}(p)| < \kappa$ (support, see Section 2.1.2 on page 5) such that for all $\gamma < \alpha$, $p \upharpoonright \gamma \in \mathbb{P}_\gamma$ and let $g_\alpha = \bigcup_{i < \alpha} g_i$. Suppose α is a successor, $\alpha = \gamma + 1$. Let $\{(\dot{S}_{\gamma j}, \delta_{\gamma j}) \mid j < \kappa\}$ be the the list of pairs as defined above. Let $(\dot{S}, \delta) = (\dot{S}_{f(\gamma)}, \delta_{f(\gamma)})$ where f is the bijection defined above. If there exists $i < \gamma$ such that $\dot{S}_{f(i)} = \dot{S}_{f(\gamma)}$ (i.e. \dot{S}_i has

been already under focus), then let $g_\alpha = g_\gamma$. Otherwise let

$$g_\alpha = g_\gamma \cup \{(\dot{S}_{f(\gamma)}, \eta)\}$$

where η is some element in $F_0 \setminus \text{ran } g_\gamma$. Doing this, we want to make sure that in the end $\text{ran } g_{\kappa^+} = F_0$. We omit the technical details needed to ensure that.

Denote $\eta = g(\dot{S}_{f(\gamma)})$. Let σ_γ be a \mathbb{P}_γ -name such that for all \mathbb{P}_γ -generic G_γ it holds that

$$\mathbb{P}_\gamma \Vdash \begin{cases} \sigma_\gamma = T(\dot{S}_{\eta+2\delta}), & \text{if } V[G_\gamma] \models [(\delta_{f(\gamma)} \in \dot{S}_{f(\gamma)}) \wedge (\dot{S}_{f(\gamma)} \text{ is stationary})] \\ \sigma_\gamma = T(\dot{S}_{\eta+2\delta+1}), & \text{if } V[G_\gamma] \models [(\delta_{f(\gamma)} \notin \dot{S}_{f(\gamma)}) \wedge (\dot{S}_{f(\gamma)} \text{ is stationary})] \\ \sigma_\gamma = \{\check{\emptyset}\}, & \text{otherwise.} \end{cases}$$

Now let \mathbb{P}_α be the collection of sequences $p = \langle \rho_i \rangle_{i \leq \gamma}$ such that $p \restriction \gamma = \langle \rho_i \rangle_{i < \gamma} \in \mathbb{P}_\gamma$, $\rho_\gamma \in \text{dom } \sigma_\gamma$ and $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \rho_\gamma \in \sigma_\gamma$ with the ordering defined in the usual way.

Let G be \mathbb{P}_{κ^+} -generic. Let us now show that the extension $V[G]$ satisfies what we want, namely that $S \subset \kappa \setminus E$ is stationary if and only if there exists $\eta \in 2^E$ such that $S = X_\eta$ (Claims 3 and 4 below).

Claim 1. For $\alpha \leq \kappa^+$ the forcing \mathbb{P}_α does not add bounded subsets of κ and the suborder

$$\mathbb{Q}_\alpha = \{p \mid p \in \mathbb{P}_\alpha, p = \langle \check{\rho}_i \rangle_{i < \alpha} \text{ where } \rho_i \in V \text{ for } i < \alpha\}$$

is dense in \mathbb{P}_α .

Proof of Claim 1. Let us show this by induction on $\alpha \leq \kappa^+$. For \mathbb{P}_0 this is already proved and the limit case is left to the reader. Suppose this is proved for all $\gamma < \alpha < \kappa^+$ and $\alpha = \beta + 1$. Then suppose $p \in \mathbb{P}_\alpha$, $p = \langle \rho_i \rangle_{i < \alpha}$. Now $p \restriction \beta \Vdash \rho_\beta \in \sigma_\beta$. Since by the induction hypothesis \mathbb{P}_β does not add bounded subsets of κ and \mathbb{Q}_β is dense in \mathbb{P}_β , there exists a condition $r \in \mathbb{Q}_\beta$, $r > p \restriction \beta$ and a standard name \check{q} such that $r \Vdash \check{q} = \rho_\beta$. Now $r \restriction \beta$ is in \mathbb{Q}_α , so it is dense in \mathbb{P}_α . To show that \mathbb{P}_α does not add bounded sets, it is enough to show that \mathbb{Q}_α does not. Let us think of \mathbb{Q}_α as a suborder of the product $\prod_{i < \alpha} 2^{<\kappa}$. Assume that τ is a \mathbb{Q}_α -name and $p \in \mathbb{Q}_\alpha$ forces that $|\tau| = \check{\lambda} < \check{\kappa}$ for some cardinal λ . Then let $\langle M_\delta \rangle_{\delta < \kappa}$ be a sequence of elementary submodels of $H(\kappa^+)$ such that for all δ, β

- (a) $|M_\delta| < \kappa$
- (b) $\delta < \beta \Rightarrow M_\delta \preceq M_\beta$,
- (c) $M_\delta \cap \kappa \subset M_\delta$,
- (d) if β is a limit ordinal, then $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$,
- (e) if $\kappa = \lambda^+$, then $M_\delta^{<\lambda} \subset M_\delta$ and if κ is inaccessible, then $M_\delta^{|M_\delta|} \subset M_{\delta+1}$,

- (f) $M_\alpha \in M_{\alpha+1}$,
- (g) $\{p, \kappa, \mathbb{Q}_\alpha, \tau, \check{E}\} \subset M_0$.

This (especially (e)) is possible since κ is not a successor of a singular cardinal and GCH holds. Now the set $C = \{M_\delta \cap \kappa \mid \delta < \kappa\}$ is cub, so because $\kappa \setminus E$ is fat, there is a closed sequence s of length $\lambda + 1$ in $C \setminus E$. Let $(\delta_i)_{i \leq \lambda}$ be the sequence such that $s = \langle M_{\delta_i} \cap \kappa \rangle_{i \leq \lambda}$. For $q \in \mathbb{Q}_\alpha$, let

$$m(q) = \inf_{\gamma \in \text{sptr } q} \text{ran } q(\gamma). \quad (\star)$$

Let $p_0 = p$ and for all $i < \gamma$ let $p_{i+1} \in M_{\delta_{i+1}} \setminus M_{\delta_i}$ be such that $p_i < p_{i+1}$, p_{i+1} decides $i + 1$ first values of τ (think of τ as a name for a function $\lambda \rightarrow \kappa$ and that p_i decides the first i values of that function) and $m(p_{i+1}) \geq M_{\delta_i} \cap \kappa$. This p_{i+1} can be found because clearly $p_i \in M_{\delta_{i+1}}$ and $M_{\delta_{i+1}}$ is an elementary submodel. If i is a limit, $i < \lambda$, then let p_i be an upper bound of $\{p_j \mid j < i\}$ which can be found in $M_{\delta_{i+1}}$ by the assumptions (f), (e) and (b), and because $M_{\delta_i} \cap \kappa \notin E$. Finally let p_λ be an upper bound of $\langle p_i \rangle_{i < \lambda}$ which exists because for all $\alpha \in \bigcup_{i < \lambda} \text{sptr } p_i$ $\sup_{i < \lambda} \text{ran } p_i(\alpha) = M_{\delta_\lambda} \cap \kappa$ is not in E and the forcing is closed under such sequences. So p_λ decides the whole τ . This completes the proof of the claim. \square Claim 1

So for simplicity, instead of \mathbb{P}_{κ^+} let us work with \mathbb{Q}_{κ^+} .

Claim 2. Let G be \mathbb{P}_{κ^+} -generic over V . Suppose $S \subset \kappa$, $S \in V[G]$ and \dot{S} is a nice name for a subset of κ such that $\dot{S}_G = S$. Then let γ be the smallest ordinal with $S \in V[G_\gamma]$. If $(S \subset \kappa \setminus E \text{ is stationary})^{V[G_\gamma]}$, then S is stationary in $V[G]$. If $\dot{S} = \dot{S}_\eta$ for some $\eta \in V$ and $V[G_\gamma] \models \sigma_\gamma \neq T((\dot{S}_\eta)_{G_\gamma \restriction \{0\}})$ for all $\gamma < \kappa^+$, then S is stationary in $V[G]$.

Proof of Claim 2. Recall, σ_γ is as in the construction of \mathbb{P}_{κ^+} . Suppose first that $S \subset \kappa \setminus E$ is a stationary set in $V[G_\gamma]$ for some $\gamma < \kappa^+$. Let us show that S is stationary in $V[G]$. Note that $V[G] = V[G_\gamma][G^\gamma]$ where $G^\gamma = G \restriction \{\alpha \mid \alpha \geq \gamma\}$. Let us show this in the case $\gamma = 0$ and $S \in V$, the other cases being similar. Let \dot{C} be a name and p a condition which forces that \dot{C} is cub. Let us show that then $p \Vdash \dot{S} \cap \dot{C} \neq \emptyset$. For $q \in \mathbb{Q}_{\kappa^+}$ let $m(q)$ be defined as in (\star) above.

Like in the proof of Claim 1, construct a continuous increasing sequence $\langle M_\alpha \rangle_{\alpha < \kappa}$ of elementary submodels of $H(\kappa^{++})$ such that $\{p, \kappa, \mathbb{P}_{\kappa^+}, \dot{S}, \dot{C}\} \subset M_0$ and $M_\alpha \cap \kappa$ is an ordinal. Since $\{M_\alpha \cap \kappa \mid \alpha < \kappa, M_\alpha \cap \kappa = \alpha\}$ is cub, there exists

$\alpha \in S$ such that $M_\alpha \cap \kappa = \alpha$ and because E does not reflect to α there exists a cub sequence

$$c \subset \{M_\beta \cap \kappa \mid \beta < \alpha, M_\beta \cap \kappa = \beta\} \setminus E,$$

$c = \langle c_i \rangle_{i < \text{cf}(\alpha)}$. Now, similarly as in the proof of Claim 1, we can choose an increasing $\langle p_i \rangle_{i \leq \text{cf}(\alpha)}$ such that $p_0 = p$, $p_i \in \mathbb{Q}_{\kappa^+}$ for all i , $p_{i+1} \Vdash \check{\beta} \in \dot{C}$ for some $c_i \leq \beta \leq c_{i+1}$, $p_{i+1} \in M_{c_{i+1}} \setminus M_{c_i}$ and $m(p_{i+1}) \geq c_i$. If i is a limit, let p_i be again an upper bound of $\{p_j \mid j < i\}$ in M_{c_i} . Since the limits are not in E , the upper bounds exist. Finally $p_{\text{cf}(\alpha)} \Vdash \alpha \in \dot{C}$, which implies $p_{\text{cf}(\alpha)} \Vdash \dot{S} \cap \dot{C} \neq \emptyset$, because α was chosen from S .

Assume then that $\dot{S} = \dot{S}_\eta$ for some $\eta \in V$ such that

$$V[G_\gamma] \models \sigma_\gamma \neq T((\dot{S}_\eta)_{G_\gamma \restriction \{0\}})$$

for all $\gamma < \kappa^+$. To prove that $(\dot{S}_\eta)_G$ is stationary in $V[G]$, we carry the same argument as the above, a little modified. Let us work in $V[G_0]$ and let p_0 force that

$$\forall \gamma < \kappa^+ (\sigma_\gamma \neq T(S_\eta)).$$

(This p_0 exists for example because there is at most one γ such that $\sigma_\gamma = T(S_\eta)$) Build the sequences c , $\langle M_{c_i} \rangle_{i < \text{cf}(\alpha)}$ and $\langle p_i \rangle_{i < \text{cf}(\alpha)}$ in the same fashion as above, except that assume additionally that the functions g_{κ^+} and f , defined along with \mathbb{P}_{κ^+} , are in M_{c_0} .

At the successor steps one has to choose p_{i+1} such that for each $\gamma \in \text{sprt } p_i$, p_{i+1} decides σ_γ . This is possible, since there are only three choices for σ_γ , namely $\{\emptyset\}$, $T(S_{\xi+2\alpha+1})$ or $T(S_{\xi+2\alpha})$ where ξ and α are justified by the functions g_{κ^+} and f . For all $\gamma \in \text{sprt } p_i$ let us denote by ξ_γ the function such that $p_{i+1} \restriction \gamma \Vdash \sigma_\gamma = T(S_{\xi_\gamma})$. Clearly $\eta \neq \xi_\gamma$ for all $\gamma \in \text{sprt } p_i$. Further demand that $m(p_{i+1}) > \sup(S_\eta \cap S_{\xi_\gamma})$ for all $\gamma \in \text{sprt } p_i$. It is possible to find such p_{i+1} from M_{i+1} because M_{i+1} is an elementary submodel and such can be found in $H(\kappa^{++})$ since $\xi_\gamma \neq \eta$ and by the definitions $S_\eta \cap S_{\xi_\gamma}$ is bounded. □_{Claim 2}

Claim 3. In $V[G]$ the following holds: if $S \subset \kappa \setminus E$ is stationary, then there exists $\eta \in 2^E$ with $\eta(0) = 0$ such that $S = X_\eta$.

Proof of Claim 3. Recall the function g_{κ^+} from the construction of \mathbb{P}_{κ^+} (defined at (**)) and the paragraph below that). Let $\eta = g_{\kappa^+}(\dot{S})$ where \dot{S} is a nice name $\dot{S} \in V$ such that $\dot{S}_G = S$. If $\alpha \in S$, then there is the smallest γ such that $\dot{S} = S_{f(\gamma)}$ and $\alpha = \delta_{f(\gamma)}$ (where f is as in the definition of \mathbb{P}_{κ^+}). This stage γ

is the only stage where it is possible that $V[G_\gamma] \models \sigma_\gamma = T(S_{\eta+2\alpha+1})$, but since $V[G_\gamma] \models \check{\alpha} \in \dot{S}$, by the definition of \mathbb{P}_{κ^+} it is not the case, so the stationarity of $S_{\eta+2\alpha+1}$ has not been killed by Claim 2. On the other hand the stationarity of $S_{\eta+2\alpha}$ is killed at this level γ of the construction, so $\alpha \in X_\eta$ by the definitions of φ and X_η . Similarly if $\alpha \notin S$, we conclude that $\alpha \notin X_\eta$. $\square_{\text{Claim 3}}$

Claim 4. In $V[G]$ the following holds: if $S \subset \kappa \setminus E$ is not stationary, then for all $\eta \in 2^E$ with $\eta(0) = 0$ we have $S \neq X_\eta$.

Proof of Claim 4. It is sufficient to show that X_η is stationary for all $\eta \in 2^E$ with $\eta(0) = 0$. Suppose first that $\eta \in F_0 \subset V$. Then since g_{κ^+} is a surjection onto F_0 (see $(***)$), there exists a name \dot{S} such that $S = \dot{S}_G$ is stationary, $S \subset \kappa \setminus E$ and $g_{\kappa^+}(S) = \eta$. Now the same argument as in the proof of Claim 3 implies that $X_\eta = S$, so X_η is stationary by Claim 2.

If $\eta \notin F_0$, then by the definition of $\eta \mapsto X_\eta$ it is sufficient to show that the \diamond -sequence added by \mathbb{P}_0 guesses in $V[G]$ every new set on a stationary set.

Suppose that τ and \dot{C} are nice \mathbb{P}_{κ^+} -names for subsets of $\check{\kappa}$ and let p be a condition forcing that \dot{C} is cub. We want to find γ and $q > p$ such that

$$q \Vdash ((\cup \dot{G}_0)(\check{\gamma}))^{-1}\{1\} = \tau \cap \check{\gamma} \wedge (\check{\gamma} \in \dot{C})$$

where $\dot{G}_0 = \dot{G} \restriction \{0\}$ is the name for the \mathbb{P}_0 -generic. To do that let $p_0 \geq p$ be such that $p_0 \Vdash \tau \notin \mathcal{P}(\check{\kappa})^V$.

Similarly as in the proofs above define a suitable sequence $\langle M_i \rangle_{i < \lambda}$ of elementary submodels, of length $\lambda < \kappa$, where λ is a cofinality of a point in E , such that $\sup_{i < \lambda} (M_i \cap \kappa) = \alpha \in E$ and $M_i \cap \kappa \notin E$ for all $i < \lambda$. Assume also that $p_0 \in M_0$. Suppose $p_i \in M_i$ is defined. Let $p_{i+1} > p_i$ be an element of $M_{i+1} \setminus M_i$ satisfying the following:

- (1) p_{i+1} decides σ_β for all $\beta \in \text{sprt } p_i$,
- (2) for all $\beta \in \text{sprt } p_i$ there is $\beta' \in M_{i+1}$ such that $p_{i+1} \Vdash \beta' \in \tau \triangle \xi_\beta$, where ξ_β is defined as in the proof of Claim 2 and p_{i+1} decides what it is,
- (3) p_{i+1} decides τ up to $M_i \cap \kappa$,
- (4) $p_{i+1} \Vdash \delta \in \dot{C}$ for some $\delta \in M_{i+1} \setminus M_i$,
- (5) $m(p_{i+1}) > M_i \cap \kappa$, ($m(p)$ is defined at (\star)),

Item (1) is possible for the same reason as in the proof of Claim 2 and (2) is possible since $p_i \Vdash \forall \eta \in \mathcal{P}(\check{\kappa})^V (\tau \neq S_{\check{\eta}})$.

Since $M_i \cap \kappa \notin E$ for $i < \lambda$, this ensures that the sequence $p_0 \leq p_1 \leq \dots$ closes under limits $< \lambda$. Let $p_\lambda = \bigcup_{i < \lambda} p_i$ and let us define $q \supset p_\lambda$ as follows: $\text{sprt } q = \text{sprt } p_\lambda$, for $\delta \in \text{sprt } p_\lambda \setminus \{0\}$ let $\text{dom } q = \alpha + 1$, $p_\lambda(\delta) \subset q(\delta)$, $q(\alpha) = 1$ and $q(0)(\alpha) = \tau \cap \gamma$ (τ means here what have been decided by $\{p_i \mid i < \lambda\}$). Now q is a condition in the forcing notion.

Now certainly, if $q \in G$, then in the extension $\tau_G \cap \alpha = (\bigcup G_0)(\alpha)^{-1}\{1\}$ and $\alpha \in C$, so we finish. $\square_{\text{Claim 4}}$

$\square_{\text{item (4)}}$

Proof of item (5). If $\kappa = \lambda^+$, this follows from the result of Mekler and Shelah [21] and Hyttinen and Rautila [10] that the existence of a $\kappa\lambda$ -canary tree is consistent. For arbitrary $\lambda < \kappa$ the result follows from the item (4) of this theorem proved above (take $Z = \kappa \setminus S_\lambda^\kappa$). $\square_{\text{item (5)}}$

Proof of item (6). For $X = \kappa$ this was proved by Halko and Shelah in [5], Theorem 4.2. For X any stationary subset of κ the proof is similar. It is sufficient to show that $2^\kappa \setminus \text{CUB}(X)$ is not meager in any open set. Suppose U is an open set and $(D_\alpha)_{\alpha < \kappa}$ is a set of dense open sets and let us show that

$$(2^\kappa \setminus \text{CUB}(X)) \cap U \cap \bigcap_{\alpha < \kappa} D_\alpha \neq \emptyset.$$

Let $p \in 2^{<\kappa}$ be such that $N_p \subset U$. Let $p_0 \geq p$ be such that $p_0 \in D_0$. Suppose p_β are defined for $\beta < \alpha + 1$. Let $p_{\alpha+1}$ be such that $p_{\alpha+1} \geq p_\alpha$, $p_{\alpha+1} \in D_{\alpha+1}$. Suppose p_β is defined for $\beta < \alpha$ and α is a limit ordinal. Let p_α be any element of $2^{<\kappa}$ such that $p_\alpha > \bigcup_{\beta < \alpha} p_\beta$, $p_\alpha(\sup_{\beta < \alpha} \text{dom } p_\beta) = 0$ and $p_\alpha \in D_\alpha$. Let $\eta = \bigcup_{\alpha < \kappa} p_\alpha$.

The complement of $\eta^{-1}\{1\}$ contains a cub, so $X \setminus \eta^{-1}\{1\}$ is stationary whence $\eta \notin \text{CUB}(X)$ and so $\eta \in 2^\kappa \setminus \text{CUB}(X)$. Also clearly $\eta \in U \cap \bigcap_{\alpha < \kappa} D_\alpha$. $\square_{\text{item (6)}}$

Proof of item (7). Our proof is different from that given by Lücke and Schlicht. Suppose $\kappa^{<\kappa} = \kappa > \omega$. We will show that in a generic extension of V all Δ_1^1 -sets have the Property of Baire. Let

$$\mathbb{P} = \{p \mid p \text{ is a function, } |p| < \kappa, \text{dom } p \subset \kappa \times \kappa^+, \text{ran } p \subset \{0, 1\}\}$$

with the ordering $p < q \iff p \subset q$ and let G be \mathbb{P} -generic over V . Suppose that $X \subset 2^\kappa$ is a Δ_1^1 -set in $V[G]$. It is sufficient to show that for every $r \in 2^{<\kappa}$

there is $q \supset r$ such that either $N_q \setminus X$ or $N_q \cap X$ is co-meager. So let $r \in 2^{<\kappa}$ be arbitrary.

Now suppose that $\langle p_i \rangle_{i < \kappa}$ and $\langle q_i \rangle_{i < \kappa}$ are sequences in $V[G]$ such that $p_i, q_i \in (2^{<\kappa})^2$ for all $i < \kappa$ and X is the projection of

$$C_0 = (2^\kappa)^2 \setminus \bigcup_{i < \kappa} N_{p_i}$$

and $2^\kappa \setminus X$ is the projection of

$$C_1 = (2^\kappa)^2 \setminus \bigcup_{i < \kappa} N_{q_i}.$$

(By N_{p_i} we mean $N_{p_i^1} \times N_{p_i^2}$ where $p_i = (p_i^1, p_i^2)$.) Since these sequences have size κ , there exists $\alpha_1 < \kappa^+$ such that they are already in $V[G_{\alpha_1}]$ where $G_{\alpha_1} = \{p \in G \mid \text{dom } p \subset \kappa \times \alpha_1\}$. More generally, for $E \subset \mathbb{P}$ and $A \subset \kappa^+$, we will denote $E_A = \{p \in E \mid \text{dom } p \subset \kappa \times A\}$ and if $p \in \mathbb{P}$, similarly $p_A = p \restriction (\kappa \times A)$.

Let $\alpha_2 \geq \alpha_1$ be such that $r \in G_{\{\alpha_2\}}$ (identifying $\kappa \times \{\alpha_2\}$ with κ). This is possible since G is generic. Let $x = G_{\{\alpha_2\}}$. Since in $V[G]$, $x \in X$ or $x \in 2^\kappa \setminus X$, there are $\alpha_3 > \alpha_2$, $p \in G_{\alpha_3}$, $p_{\{\alpha_2\}} \supset r$ and a name τ such that p forces that $(x, \tau) \notin N_{p_i}$ for all $i < \kappa$ or $(x, \tau) \notin N_{q_i}$ for all $i < \kappa$. Without loss of generality assume that p forces that $(x, \tau) \notin N_{p_i}$ for all $i < \kappa$. Also we can assume that τ is a \mathbb{P}_{α_3} -name and that $\alpha_3 = \alpha_2 + 2$.

By working in $V[G_{\alpha_2}]$ we may assume that $\alpha_2 = 0$. For all $q \in \mathbb{P}_{\{1\}}$, $p_{\{1\}} \subseteq q$ and $i < \kappa$, let $D_{i,q}$ be the set of all $s \in \mathbb{P}_{\{0\}}$ such that $p_{\{0\}} \subseteq s$, $\text{dom}(s) \geq \text{dom}(p_i^1)$ and there is $q' \in \mathbb{P}_{\{1\}}$ such that $q \subseteq q'$ and $s \cup q'$ decides $\tau \restriction \text{dom}(p_i^2)$. Clearly each $D_{i,q}$ is dense above $p_{\{0\}}$ in $\mathbb{P}_{\{0\}}$ and thus it is enough to show that if $y \in 2^\kappa$ is such that for all $i < \kappa$ and q as above there is $\alpha < \kappa$ such that $y \restriction \alpha \in D_{i,q}$, then $y \in X$.

So let y be such. Then we can find $z \in 2^\kappa$ such that for all $i < \kappa$ and q as above there are $\alpha, \beta < \kappa$ such that $\alpha \geq \text{dom}(p_i^1)$ and $y \restriction \alpha \cup z \restriction \beta$ decides $t = \tau \restriction \text{dom}(p_i^2)$. By the choice of p , $(y \restriction \text{dom}(p_i^1), t) \neq p_i$. Thus letting τ^* be the function as decided by y and z , $(y, \tau^*) \in C_0$ and thus $y \in X$. □_{item (7)}

□ Theorem 49

Remark ($\text{cf}(\kappa) = \kappa > \omega$). There are some more results and strengthenings of the results in Theorem 49:

- (1) (Independently known by S. Coskey and P. Schlicht) If $V = L$ then there is a Δ_1^1 wellorder of $\mathcal{P}(\kappa)$ and this implies that there is a Δ_1^1 set without the Baire Property.

- (2) Suppose that $\omega < \kappa < \lambda$, κ regular and λ inaccessible. Then after turning λ into κ^+ by collapsing each ordinal less than λ to κ using conditions of size $< \kappa$, the Baire Property holds for Δ_1^1 subsets of κ^κ .

50. Corollary. *For a regular $\lambda < \kappa$ let NS_λ denote the equivalence relation on 2^κ such that $\eta \text{NS}_\lambda \xi$ if and only if $\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is not λ -stationary. Then NS_λ is not Borel and it is not Δ_1^1 in L or in the forcing extensions after adding κ^+ Cohen subsets of κ .*

Proof. Define a map $f: 2^\kappa \rightarrow (2^\kappa)^2$ by $\eta \mapsto (\emptyset, \kappa \setminus \eta)$. Suppose for a contradiction that NS_λ is Borel. Then

$$\text{NS}_\emptyset = \text{NS}_\lambda \cap \underbrace{\{(\emptyset, \eta) \mid \eta \in 2^\kappa\}}_{\text{closed}}$$

is Borel, and further $f^{-1}[\text{NS}_\emptyset]$ is Borel by continuity of f . But $f^{-1}[\text{NS}_\emptyset]$ equals CUB which is not Borel by Theorem 49 (6) and Theorem 45. Similarly, using items (2) and (3) of Theorem 49, one can show that NS_λ is not Δ_1^1 under the stated assumptions. \square

4.4. The Partial Orders $\langle \mathcal{E}, \leq_B \rangle$. Let $\mathcal{C} \in \{\text{Borel}, \Delta_1^1, \text{Borel}^*, \Sigma_1^1\}$ and define

$$\mathcal{E}_\kappa^\mathcal{C} = \{E \subset 2^\kappa \times 2^\kappa \mid (E \in \mathcal{C}) \wedge (E \text{ is an equivalence relation})\}.$$

Equip $\mathcal{E}_\kappa^\mathcal{C}$ with the partial order \leq_B . In the case $\kappa = \omega$ there are many known results that describe the order $\langle \mathcal{E}_\omega^{\text{Borel}}, \leq_B \rangle$. Some results were discussed in Section 4.2.3, some other results show that this order is very complicated. To mention two:

51. Theorem (Louveau-Velickovic [20]). *The partial order $\langle \mathcal{P}(\omega), \subset_* \rangle$ can be embedded into $\langle \mathcal{E}_\omega^{\text{Borel}}, \leq_B \rangle$, where $A \subset_* B$ if $A \setminus B$ is finite.*

52. Theorem (Adams-Kechris [1]). *The partial order $\langle \mathcal{B}, \subset \rangle$ can be embedded into $\langle \mathcal{E}_\omega^{\text{Borel}}, \leq_B \rangle$, where \mathcal{B} is the collection of all Borel subsets of the real line \mathbb{R} . In fact, the embedding is into the suborder of $\langle \mathcal{E}_\omega^{\text{Borel}}, \leq_B \rangle$ consisting of the countable Borel equivalence relations, i.e., those Borel equivalence relations each of whose equivalence classes is countable.*

4.4.1. An Embedding of $\langle \mathcal{P}(\kappa), \subset \rangle$ into $\langle \mathcal{E}, \leq_B \rangle$. In this section we aim to prove the following weak version of such a theorem for $\kappa > \omega$:

53. Theorem. *Suppose GCH and κ is regular, uncountable. Then in a cofinality and GCH preserving forcing extension, there is an embedding*

$$\langle \mathcal{P}(\kappa), \subset \rangle \rightarrow \langle \mathcal{E}^{Borel^*}, \leq_B \rangle.$$

Proof. In this proof we identify functions $\eta \in 2^{\leq \kappa}$ with the sets $\eta^{-1}\{1\}$: for example we write $\eta \cap \xi$ to mean $\eta^{-1}\{1\} \cap \xi^{-1}\{1\}$.

The equivalence relations in the range of the embedding will have the following form. For $X \subset S_\mu^\kappa$, we denote by E_X the relation

$$E_X = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}) \cap X \text{ is not stationary}\}.$$

This relation is easily seen to be Σ_1^1 . If $\mu = \omega$, then it is in fact Borel*. To see this use the same argument as in the proof of Theorem 49 (1) that the CUB_ω^κ set is Borel*. So, we will carry out the argument for an almost arbitrary regular cardinal $\mu < \kappa$ where “almost” means: if $\kappa = \lambda^+$ and λ is singular, then we demand that $\mu \leq \text{cf}(\lambda)$. The statement of the theorem then follows putting $\mu = \omega$.

The embedding will look as follows. Let $(S_i)_{i < \kappa}$ be pairwise disjoint stationary subsets of

$$\lim S_\mu^\kappa = \{\alpha \in S_\mu^\kappa \mid \alpha \text{ is a limit of ordinals in } S_\mu^\kappa\}. \quad (*)$$

Denote

$$K(A) = E \bigcup_{\alpha \in A} S_\alpha. \quad (**)$$

We intend that $A \mapsto K(A)$ is the embedding. If $X_1 \subset X_2 \subset \kappa$, then $E_{X_1} \leq_B E_{X_2}$, because $f(\eta) = \eta \cap X_1$ is a reduction. This guarantees that

$$A_1 \subset A_2 \implies K(A_1) \leq_B K(A_2).$$

Now suppose that for all $\alpha < \kappa$ we have killed (by forcing) all reductions from $K(\alpha) = E_{S_\alpha}$ to $K(\kappa \setminus \alpha) = E_{\bigcup_{\beta \neq \alpha} S_\beta}$ for all $\alpha < \kappa$. Then if $K(A_1) \leq_B K(A_2)$ it follows that $A_1 \subset A_2$: Otherwise choose $\alpha \in A_1 \setminus A_2$ and we have:

$$K(\alpha) \leq_B K(A_1) \leq_B K(A_2) \leq_B K(\kappa \setminus \alpha),$$

contradiction. So we have:

$$A_1 \subset A_2 \iff K(A_1) \leq_B K(A_2)$$

and therefore K is the desired embedding.

Suppose that $f: E_X \leq_B E_Y$ is a Borel reduction. Then $g: 2^\kappa \rightarrow 2^\kappa$ defined by $g(\eta) = f(\eta) \triangle f(0)$ is a Borel function with the following property:

$$\eta \cap X \text{ is stationary} \iff g(\eta) \cap Y \text{ is stationary.}$$

The function g is Borel, so by Lemma 33 there are dense open sets D_i for $i < \kappa$ such that $g \upharpoonright D$ is continuous where $D = \bigcap_{i < \kappa} D_i$. Note that D_i are open so for each i we can write $D_i = \bigcup_{j < \kappa} N_{p(i,j)}$, where $(p(i,j))_{j < \kappa}$ is a suitable collection of elements of $2^{<\kappa}$.

Next define $Q_g: 2^{<\kappa} \times 2^{<\kappa} \rightarrow \{0, 1\}$ by $Q_g(p, q) = 1 \iff N_p \cap D \subset g^{-1}[N_q]$ and $R_g: \kappa \times \kappa \rightarrow 2^{<\kappa}$ by $R_g(i, j) = p(i, j)$ where $p(i, j)$ are as above.

For any $Q: 2^{<\kappa} \times 2^{<\kappa} \rightarrow \{0, 1\}$ define $Q^*: 2^\kappa \rightarrow 2^\kappa$ by

$$Q^*(\eta) = \begin{cases} \xi, & \text{s.t. } \forall \alpha < \kappa \exists \beta < \kappa Q(\eta \upharpoonright \beta, \xi \upharpoonright \alpha) = 1 \text{ if such exists,} \\ 0, & \text{otherwise.} \end{cases}$$

And for any $R: \kappa \times \kappa \rightarrow 2^{<\kappa}$ define

$$R^* = \bigcap_{i < \kappa} \bigcup_{j < \kappa} N_{R(i,j)}.$$

Now clearly $R_g^* = D$ and $Q_g^* \upharpoonright D = g \upharpoonright D$, i.e. (Q, D) codes $g \upharpoonright D$ in this sense. Thus we have shown that if there is a reduction $E_X \leq_B E_Y$, then there is a pair (Q, R) which satisfies the following conditions:

- (1) $Q: (2^{<\kappa})^2 \rightarrow \{0, 1\}$ is a function.
- (2) $Q(\emptyset, \emptyset) = 1$.
- (3) If $Q(p, q) = 1$ and $p' > p$, then $Q(p', q) = 1$.
- (4) If $Q(p, q) = 1$ and $q' < q$, then $Q(p, q') = 1$.
- (5) Suppose $Q(p, q) = 1$ and $\alpha > \text{dom } q$. There exist $q' > q$ and $p' > p$ such that $\text{dom } q' = \alpha$ and $Q(p', q') = 1$.
- (6) If $Q(p, q) = Q(p, q') = 1$, then $q \leq q'$ or $q' < q$.
- (7) $R: \kappa \times \kappa \rightarrow 2^{<\kappa}$ is a function.
- (8) For each $i \in \kappa$ the set $\bigcup_{j < \kappa} N_{R(i,j)}$ is dense.
- (9) For all $\eta \in R^*$, $\eta \cap X$ is stationary if and only if $Q^*(\eta \cap X) \cap Y$ is stationary.

Let us call a pair (Q, R) which satisfies (1)–(9) a *code for a reduction (from E_X to E_Y)*. Note that it is not the same as the Borel code for the graph of a reduction function as a set. Thus we have shown that if $E_X \leq_B E_Y$, then there exists a code for a reduction from E_X to E_Y . We will now prove the following

lemma which is stated in a general enough form so we can use it also in the next section:

54. Lemma (GCH). *Suppose μ_1 and μ_2 are regular cardinals less than κ such that if $\kappa = \lambda^+$, then $\mu_2 \leq \text{cf}(\lambda)$, and suppose X is a stationary subset of $S_{\mu_1}^\kappa$, Y is a subset of $S_{\mu_2}^\kappa$, $X \cap Y = \emptyset$ (relevant if $\mu_1 = \mu_2$) and if $\mu_1 < \mu_2$ then $\alpha \cap X$ is not stationary in α for all $\alpha \in Y$. Suppose that (Q, R) is an arbitrary pair. Denote by φ the statement “ (Q, R) is not a code for a reduction from E_X to E_Y ”. Then there is a κ^+ -c.c. $< \kappa$ -closed forcing \mathbb{R} such that $\mathbb{R} \Vdash \varphi$.*

Remark. Clearly if $\mu_1 = \mu_2 = \omega$, then the condition $\mu_2 \leq \text{cf}(\lambda)$ is of course true. We need this assumption in order to have $\nu^{<\mu_2} < \kappa$ for all $\nu < \kappa$.

Proof of Lemma 54. We will show that one of the following holds:

- (1) φ already holds, i.e. $\{\emptyset\} \Vdash \varphi$,
- (2) $\mathbb{P} = 2^{<\kappa} = \{p: \alpha \rightarrow 2 \mid \alpha < \kappa\} \Vdash \varphi$,
- (3) $\mathbb{R} \Vdash \varphi$,

where

$$\mathbb{R} = \{(p, q) \mid p, q \in 2^\alpha, \alpha < \kappa, X \cap p \cap q = \emptyset, q \text{ is } \mu_1\text{-closed}\}.$$

Above “ q is μ_1 -closed” means “ $q^{-1}\{1\}$ is μ_1 -closed” etc., and we will use this abbreviation below. Assuming that (1) and (2) do not hold, we will show that (3) holds.

Since (2) does not hold, there is a $p \in \mathbb{P}$ which forces $\neg\varphi$ and so $\mathbb{P}_p = \{q \in \mathbb{P} \mid q > p\} \Vdash \neg\varphi$. But $\mathbb{P}_p \cong \mathbb{P}$, so in fact $\mathbb{P} \Vdash \neg\varphi$, because φ has only standard names as parameters (names for elements in V , such as Q , R , X and Y). Let G be any \mathbb{P} -generic and let us denote the set $G^{-1}\{1\}$ also by G . Let us show that $G \cap X$ is stationary. Suppose that \dot{C} is a name and $r \in \mathbb{P}$ is a condition which forces that \dot{C} is cub. For an arbitrary q_0 , let us find a $q > q_0$ which forces $\dot{C} \cap \dot{G} \cap \check{X} \neq \emptyset$. Make a counter assumption: no such $q > q_0$ exists. Let $q_1 > q_0$ and $\alpha_1 > \text{dom } q_0$ be such that $q_1 \Vdash \check{\alpha}_1 \in \dot{C}$, $\text{dom } q_1 > \alpha_1$ is a successor and $q_1(\max \text{dom } q_1) = 1$. Then by induction on $i < \kappa$ let q_{i+1} and $\alpha_{i+1} > \text{dom } q_i$ be such that $q_{i+1} \Vdash \check{\alpha}_{i+1} \in \dot{C}$, $\text{dom } q_{i+1} > \alpha_{i+1}$ is a successor and $q_{i+1}(\max \text{dom } q_{i+1}) = 1$. If j is a limit ordinal, let $q_j = \bigcup_{i < j} q_i \cup \{(\sup_{i < j} \text{dom } q_i, 1)\}$ and $\alpha_j = \sup_{i < j} \alpha_i$. We claim that for some $i < \kappa$, the condition q_i is as needed, i.e.

$$q_i \Vdash \dot{G} \cap \check{X} \cap \dot{C} \neq \emptyset.$$

Clearly for limit ordinals j , we have $\alpha_j = \max \text{dom } q_j$ and $q_j(\alpha_j) = 1$ and $\{\alpha_j \mid j \text{ limit}\}$ is cub. Since X is stationary, there exists a limit j_0 such that $\alpha_{j_0} \in X$. Because q_0 forces that \dot{C} is cub, $q_j > q_i > q_0$ for all $i < j$, $q_i \Vdash \check{\alpha}_i \in \dot{C}$ and $\alpha_j = \sup_{i < j} \alpha_i$, we have $q_j \Vdash \alpha_j \in \dot{C} \cap \check{X}$. On the other hand $q_j(\alpha_j) = 1$, so $q_j \Vdash \alpha_j \in G$ so we finish.

So now we have in $V[G]$ that $G \cap X$ is stationary, $G \in R^*$ (since R^* is co-meager) and Q is a code for a reduction, so Q^* has the property (9) and $Q^*(G \cap X) \cap Y$ is stationary. Denote $Z = Q^*(G \cap X) \cap Y$. We will now construct a forcing \mathbb{Q} in $V[G]$ such that

$$V[G] \models (\mathbb{Q} \Vdash "G \cap X \text{ is not stationary, but } Z \text{ is stationary}").$$

Then $V[G] \models (\mathbb{Q} \Vdash \varphi)$ and hence $\mathbb{P} * \mathbb{Q} \Vdash \varphi$. On the other hand \mathbb{Q} will be chosen such that $\mathbb{P} * \mathbb{Q}$ and \mathbb{R} give the same generic extensions. So let

$$\mathbb{Q} = \{q: \alpha \rightarrow 2 \mid X \cap G \cap q = \emptyset, q \text{ is } \mu_1\text{-closed}\}, \quad (***)$$

Clearly \mathbb{Q} kills the stationarity of $G \cap X$. Let us show that it preserves the stationarity of Z . For that purpose it is sufficient to show that for any nice \mathbb{Q} -name \dot{C} for a subset of κ and any $p \in \mathbb{Q}$, if $p \Vdash "\dot{C} \text{ is } \mu_2\text{-cub}"$, then $p \Vdash (\dot{C} \cap \check{Z} \neq \check{\emptyset})$.

So suppose \dot{C} is a nice name for a subset of κ and $p \in \mathbb{Q}$ is such that

$$p \Vdash "\dot{C} \text{ is cub}."$$

Let $\lambda > \kappa$ be a sufficiently large regular cardinal and let N be an elementary submodel of $\langle H(\lambda), p, \dot{C}, \mathbb{Q}, \kappa \rangle$ which has the following properties:

- $|N| = \mu_2$,
- $N^{<\mu_2} \subset N$,
- $\alpha = \sup(N \cap \kappa) \in Z$ (This is possible because Z is stationary).

Here we use the hypothesis that μ_2 is at most $\text{cf}(\lambda)$ when $\kappa = \lambda^+$. Now by the assumption of the theorem, $\alpha \setminus X$ contains a μ_1 -closed unbounded sequence of length μ_2 , $\langle \alpha_i \rangle_{i < \mu_2}$. Let $\langle D_i \rangle_{i < \mu_2}$ list all the dense subsets of \mathbb{Q}^N in N . Let $q_0 \geq p$, $q_0 \in \mathbb{Q}^N$ be arbitrary and suppose $q_i \in \mathbb{Q}^N$ is defined for all $i < \gamma$. If $\gamma = \beta + 1$, then define q_γ to be an extension of q_β such that $q_\gamma \in D_\beta$ and $\text{dom } q_\gamma = \alpha_i$ for some $\alpha_i > \text{dom } q_\beta$. To do that, for instance, choose $\alpha_i > \text{dom } q_\beta$ and define $q' \supset q_\beta$ by $\text{dom } q' = \alpha_i$, $q(\delta) = 0$ for all $\delta \in \text{dom } q' \setminus \text{dom } q_\beta$ and then extend q' to q_β in D_β . If γ is a limit ordinal with $\text{cf}(\gamma) \neq \mu_1$, then let $q_\gamma = \bigcup_{i < \gamma} q_i$. If

$\text{cf}(\gamma) = \mu_1$, let

$$q_\gamma = \left(\bigcup_{i < \gamma} q_i \right) \frown \langle \sup_{i < \gamma} \text{dom } q_i, 1 \rangle.$$

Since N is closed under taking sequences of length less than μ_2 , $q_\gamma \in N$. Since we required elements of \mathbb{Q} to be μ_1 -closed but not γ -closed if $\text{cf}(\gamma) \neq \mu_1$, $q_\gamma \in \mathbb{Q}$ when $\text{cf}(\gamma) \neq \mu_1$. When $\text{cf}(\gamma) = \mu_1$, the limit $\sup_{i < \gamma} \text{dom } q_i$ coincides with a limit of a subsequence of $\langle \alpha_i \rangle_{i < \mu_2}$ of length μ_1 , i.e. the limit is α_β for some β since this sequence is μ_1 -closed. So by definition $\sup_{i < \gamma} \text{dom } q_i \notin X$ and again $q_\gamma \in \mathbb{Q}$.

Then $q = \bigcup_{\gamma < \mu} q_\gamma$ is a \mathbb{Q}^N -generic over N . Since $X \cap Y = \emptyset$, also $(X \cap G) \cap Z = \emptyset$ and $\alpha \notin X \cap G$. Hence $q \frown (\alpha, 1)$ is in \mathbb{Q} . We claim that $q \Vdash (\dot{C} \cap \check{Z} \neq \emptyset)$.

Because $p \Vdash \dot{C}$ is unbounded, also $N \models (p \Vdash \dot{C} \text{ is unbounded})$ by elementarity. Assuming that λ is chosen large enough, we may conclude that for all \mathbb{Q}^N -generic g over N , $N[g] \models \dot{C}_g$ is unbounded, thus in particular $N[g] \models \dot{C}_g$ is unbounded in κ . Let G_1 be \mathbb{Q} -generic over $V[G]$ with $q \in G_1$. Then $\dot{C}_{G_1} \supset \dot{C}_q$ which is unbounded in α by the above, since $\sup(\kappa \cap N) = \alpha$. Because \dot{C}_{G_1} is μ_2 -cub, α is in \dot{C}_{G_1} .

Thus $\mathbb{P} * \mathbb{Q} \Vdash \varphi$. It follows straightforwardly from the definition of iterated forcing that \mathbb{R} is isomorphic to a dense suborder of $\mathbb{P} * \dot{\mathbb{Q}}$ where $\dot{\mathbb{Q}}$ is a \mathbb{P} -name for a partial order such that $\dot{\mathbb{Q}}_G$ equals \mathbb{Q} as defined in $(***)$ for any \mathbb{P} -generic G .

Now it remains to show that \mathbb{R} has the κ^+ -c.c. and is $< \kappa$ -closed. Since \mathbb{R} is a suborder of $\mathbb{P} \times \mathbb{P}$, which has size κ , it trivially has the κ^+ -c.c. Suppose $(p_i, q_i)_{i < \gamma}$ is an increasing sequence, $\gamma < \kappa$. Then the pair

$$(p, q) = \left\langle \left(\bigcup_{i < \gamma} p_i \right) \frown \langle \alpha, 0 \rangle, \left(\bigcup_{i < \gamma} q_i \right) \frown \langle \alpha, 1 \rangle \right\rangle$$

is an upper bound. □ Lemma 54

Remark. Note that the forcing used in the previous proof is equivalent to κ -Cohen forcing.

55. Corollary (GCH). *Let $K: A \mapsto E_{\bigcup_{\alpha \in A} S_\alpha}$ be as in the beginning of the proof. For each pair (Q, R) and each α there is a $< \kappa$ -closed, κ^+ -c.c. forcing $\mathbb{R}(Q, R, \alpha)$ such that*

$\mathbb{R}(Q, R, \alpha) \Vdash \text{“}(Q, R) \text{ is not a code for a reduction from } K(\{\alpha\}) \text{ to } K(\kappa \setminus \{\alpha\})\text{”}$

Proof. By the above lemma one of the choices $\mathbb{R} = \{\emptyset\}$, $\mathbb{R} = 2^{<\kappa}$ or

$$\mathbb{R} = \{(p, q) \mid p, q \in 2^\beta, \beta < \kappa, S_\alpha \cap p \cap q = \emptyset, q \text{ is } \mu\text{-closed}\}$$

suffices. □

Start with a model satisfying GCH. Let $h: \kappa^+ \rightarrow \kappa^+ \times \kappa \times \kappa^+$ be a bijection such that $h_3(\alpha) < \alpha$ for $\alpha > 0$ and $h_3(0) = 0$. Let $\mathbb{P}_0 = \{\emptyset\}$. For each $\alpha < \kappa$, let $\{\sigma_{\beta\alpha 0} \mid \beta < \kappa^+\}$ be the list of all \mathbb{P}_0 -names for codes for a reduction from $K(\{\alpha\})$ to $K(\kappa \setminus \{\alpha\})$. Suppose \mathbb{P}_i and $\{\sigma_{\beta\alpha i} \mid \beta < \kappa^+\}$ are defined for all $i < \gamma$ and $\alpha < \kappa$, where $\gamma < \kappa^+$ is a successor $\gamma = \beta + 1$, \mathbb{P}_i is $< \kappa$ -closed and has the κ^+ -c.c.

Consider $\sigma_{h(\beta)}$. By the above corollary, the following holds:

$$\begin{aligned} \mathbb{P}_\beta \Vdash [\exists \mathbb{R} \in \mathcal{P}(2^{<\kappa} \times 2^{<\kappa}) (\mathbb{R} \text{ is } < \kappa\text{-closed, } \kappa^+\text{-c.c. p.o. and} \\ \mathbb{R} \Vdash \text{“}\sigma_{h(\beta)} \text{ is not a code for a reduction.”})] \end{aligned}$$

So there is a \mathbb{P}_β -name ρ_β such that \mathbb{P}_β forces that ρ_β is as \mathbb{R} above. Define

$$\mathbb{P}_\gamma = \{(p_i)_{i<\gamma} \mid ((p_i)_{i<\beta} \in \mathbb{P}_\beta) \wedge ((p_i)_{i<\beta} \Vdash p_\beta \in \rho_\beta)\}.$$

And if $p = (p_i)_{i<\gamma} \in \mathbb{P}_\gamma$ and $p' = (p'_i)_{i<\gamma} \in \mathbb{P}_\gamma$, then

$$p \leq_{\mathbb{P}_\gamma} p' \iff [(p_i)_{i<\beta} \leq_{\mathbb{P}_\beta} (p'_i)_{i<\beta}] \wedge [(p'_i)_{i<\beta} \Vdash (p_\beta \leq_{\rho_\beta} p'_\beta)].$$

If γ is a limit, $\gamma \leq \kappa^+$, let

$$\mathbb{P}_\gamma = \{(p_i)_{i<\gamma} \mid \forall \beta (\beta < \gamma \rightarrow (p_i)_{i<\beta} \in \mathbb{P}_\beta) \wedge (|\text{sprt}(p_i)_{i<\gamma}| < \kappa)\},$$

where sprt means support, see Section 2.1.2 on page 5. For every α , let $\{\sigma_{\beta\alpha\gamma} \mid \beta < \kappa^+\}$ list all \mathbb{P}_β -names for codes for a reduction. It is easily seen that \mathbb{P}_γ is $< \kappa$ -closed and has the κ^+ -c.c. for all $\gamma \leq \kappa^+$.

We claim that \mathbb{P}_{κ^+} forces that for all α , $K(\{\alpha\}) \not\leq_B K(\kappa \setminus \{\alpha\})$ which suffices by the discussion in the beginning of the proof, see $(**)$ for the notation.

Let G be \mathbb{P}_{κ^+} -generic and let $G_\gamma = “G \cap \mathbb{P}_\gamma”$ for every $\gamma < \kappa$. Then G_γ is \mathbb{P}_γ -generic.

Suppose that in $V[G]$, $f: 2^\kappa \rightarrow 2^\kappa$ is a reduction $K(\{\alpha\}) \leq_B K(\kappa \setminus \{\alpha\})$ and (Q, R) is the corresponding code for a reduction. By [19] Theorem VIII.5.14, there is a $\delta < \kappa^+$ such that $(Q, R) \in V[G_\delta]$. Let δ_0 be the smallest such δ .

Now there exists $\sigma_{\gamma\alpha\delta_0}$, a \mathbb{P}_{δ_0} -name for (Q, R) . By the definition of h , there exists a $\delta > \delta_0$ with $h(\delta) = (\gamma, \alpha, \delta_0)$. Thus

$$\mathbb{P}_{\delta+1} \Vdash \text{“}\sigma_{\gamma\alpha\delta_0} \text{ is not a code for a reduction”},$$

i.e. $V[G_{\delta+1}] \models (Q, R)$ is not a code for a reduction. Now one of the items (1)–(9) fails for (Q, R) in $V[G_{\delta+1}]$. We want to show that then one of them fails in $V[G]$. The conditions (1)–(8) are absolute, so if one of them fails in $V[G_{\delta+1}]$, then we are done. Suppose (1)–(8) hold but (9) fails. Then there is an $\eta \in R^*$ such that

$Q^*(\eta \cap S_{\{\alpha\}}) \cap S_{\kappa \setminus \alpha}$ is stationary but $\eta \cap S_{\{\alpha\}}$ is not or vice versa. In $V[G_{\delta+1}]$ define

$$\mathbb{P}^{\delta+1} = \{(p_i)_{i < \kappa^+} \in \mathbb{P}_{\kappa^+} \mid (p_i)_{i < \delta+1} \in G_{\delta+1}\}.$$

Then $\mathbb{P}^{\delta+1}$ is $< \kappa$ -closed. Thus it does not kill stationarity of any set. So if $G^{\delta+1}$ is $\mathbb{P}_{\delta+1}$ -generic over $V[G_{\delta+1}]$, then in $V[G_{\delta+1}][G^{\delta+1}]$, (Q, R) is not a code for a reduction. Now it remains to show that $V[G] = V[G_{\delta+1}][G^{\delta+1}]$ for some $G^{\delta+1}$. In fact putting $G^{\delta+1} = G$ we get $\mathbb{P}^{\delta+1}$ -generic over $V[G_{\delta+1}]$ and of course $V[G_{\delta+1}][G] = V[G]$ (since $G_{\delta+1} \subset G$). \square Theorem 53

Remark. The forcing constructed in the proof of Theorem 53 above, combined with the forcing in the proof of item (4) of Theorem 49 gives that for $\kappa^{<\kappa} = \kappa > \omega_1$ not successor of a singular cardinal, we have in a forcing extension that $\langle \mathcal{P}(\kappa), \subset \rangle$ embeds into $\langle \mathcal{E}^{\Delta_1^1}, \leq_B \rangle$, i.e. the partial order of Δ_1^1 -equivalence relations under Borel reducibility.

Open Problem. Can there be two equivalence relations, E_1 and E_2 on 2^κ , $\kappa > \omega$ such that E_1 and E_2 are Borel and incomparable, i.e. $E_1 \not\leq_B E_2$ and $E_2 \not\leq_B E_1$?

4.4.2. *Reducibility Between Different Cofinalities.* Recall the notation defined in Section 2.1. In this section we will prove the following two theorems:

56. Theorem. *Suppose that κ is a weakly compact cardinal and that $V = L$. Then*

- (A) $E_{S_\lambda^\kappa} \leq_c E_{\text{reg}(\kappa)}$ for any regular $\lambda < \kappa$, where $\text{reg}(\kappa) = \{\lambda < \kappa \mid \lambda \text{ is regular}\}$,
- (B) In a forcing extension $E_{S_\omega^{\omega_2}} \leq_c E_{S_{\omega_1}^{\omega_2}}$. Similarly for λ , λ^+ and λ^{++} instead of ω , ω_1 and ω_2 for any regular $\lambda < \kappa$.

57. Theorem. *For a cardinal κ which is a successor of a regular cardinal or κ inaccessible, there is a cofinality-preserving forcing extension in which for all regular $\lambda < \kappa$, the relations $E_{S_\lambda^\kappa}$ are \leq_B -incomparable with each other.*

Let us begin by proving the latter.

Proof of Theorem 57. Let us show that there is a forcing extension of L in which $E_{S_{\omega_1}^{\omega_2}}$ and $E_{S_\omega^{\omega_2}}$ are incomparable. The general case is similar.

We shall use Lemma 54 with $\mu_1 = \omega$ and $\mu_2 = \omega_1$ and vice versa, and then a similar iteration as in the end of the proof of Theorem 53. First we force, like in the proof of Theorem 49 (4), a stationary set $S \subset S_\omega^{\omega_2}$ such that for all $\alpha \in S_{\omega_1}^{\omega_2}$, $\alpha \cap S$ is non-stationary in α . Also for all $\alpha \in S_\omega^{\omega_2}$, $\alpha \cap S_{\omega_1}^{\omega_2}$ is non-stationary.

By Lemma 54, for each code for a reduction from E_S to $E_{S_{\omega_1}^{\omega_2}}$ there is a $< \omega_2$ -closed ω_3 -c.c. forcing which kills it. Similarly for each code for a reduction from $E_{S_{\omega_1}^{\omega_2}}$ to $E_{S_{\omega_1}^{\omega_2}}$. Making an ω_3 -long iteration, similarly as in the end of the proof of Theorem 53, we can kill all codes for reductions from E_S to $E_{S_{\omega_1}^{\omega_2}}$ and from $E_{S_{\omega_1}^{\omega_2}}$ to $E_{S_{\omega_1}^{\omega_2}}$. Thus, in the extension there are no reductions from $E_{S_{\omega_1}^{\omega_2}}$ to $E_{S_{\omega_1}^{\omega_2}}$ and no reductions from $E_{S_{\omega_1}^{\omega_2}}$ to $E_{S_{\omega_1}^{\omega_2}}$. (Suppose there is one of a latter kind, $f: 2^{\omega_2} \rightarrow 2^{\omega_2}$. Then $g(\eta) = f(\eta \cap S)$ is a reduction from E_S to $E_{S_{\omega_1}^{\omega_2}}$.) $\square_{\text{Theorem 57}}$

58. Definition. Let X, Y be subsets of κ and suppose Y consists of ordinals of uncountable cofinality. We say that X \diamond -reflects to Y if there exists a sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ such that

- (1) $D_\alpha \subset \alpha$ is stationary in α ,
- (2) if $Z \subset X$ is stationary, then $\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}$ is stationary.

59. Theorem. If X \diamond -reflects to Y , then $E_X \leq_c E_Y$.

Proof. Let $\langle D_\alpha \rangle_{\alpha \in Y}$ be the sequence of Definition 58. For a set $A \subset \kappa$ define

$$f(A) = \{\alpha \in Y \mid A \cap X \cap D_\alpha \text{ is stationary in } \alpha\}. \quad (i)$$

We claim that f is a continuous reduction. Clearly f is continuous. Assume that $(A \triangle B) \cap X$ is non-stationary. Then there is a cub set $C \subset \kappa \setminus [(A \triangle B) \cap X]$. Now $A \cap X \cap C = B \cap X \cap C$ (ii). The set $C' = \{\alpha < \kappa \mid C \cap \alpha \text{ is unbounded in } \alpha\}$ is also cub and if $\alpha \in Y \cap C'$, we have that $D_\alpha \cap C$ is stationary in α . Therefore for $\alpha \in Y \cap C'$ (iii) we have the following equivalences:

$$\begin{aligned} \alpha \in f(A) &\iff A \cap X \cap D_\alpha \text{ is stationary} \\ &\stackrel{(iii)}{\iff} A \cap X \cap C \cap D_\alpha \text{ is stationary} \\ &\stackrel{(ii)}{\iff} B \cap X \cap C \cap D_\alpha \text{ is stationary} \\ &\stackrel{(iii)}{\iff} B \cap X \cap D_\alpha \text{ is stationary} \\ &\stackrel{(i)}{\iff} \alpha \in f(B) \end{aligned}$$

Thus $(f(A) \triangle f(B)) \cap Y \subset \kappa \setminus C'$ and is non-stationary.

Suppose $A \triangle B$ is stationary. Then either $A \setminus B$ or $B \setminus A$ is stationary. Without loss of generality suppose the former. Then

$$S = \{\alpha \in Y \mid (A \setminus B) \cap X \cap \alpha = D_\alpha\}$$

is stationary by the definition of the sequence $\langle D_\alpha \rangle_{\alpha \in Y}$. Thus for $\alpha \in S$ we have that $A \cap X \cap D_\alpha = A \cap X \cap (A \setminus B) \cap X \cap \alpha = (A \setminus B) \cap X \cap \alpha$ is stationary in α and $B \cap X \cap D_\alpha = B \cap X \cap (A \setminus B) \cap X \cap \alpha = \emptyset$ is not stationary in α . Therefore $(f(A) \triangle f(B)) \cap Y$ is stationary (as it contains S). \square

Fact (Π_1^1 -reflection). *Assume that κ is weakly compact. If R is any binary predicate on V_κ and $\forall A \varphi$ is some Π_1^1 -sentence where φ is a first-order sentence in the language of set theory together with predicates $\{R, A\}$ such that $(V_\kappa, R) \models \forall A \varphi$, then there exists stationary many $\alpha < \kappa$ such that $(V_\alpha, R \cap V_\alpha) \models \forall A \varphi$.*

We say that X *strongly reflects to* Y if for all stationary $Z \subset X$ there exist stationary many $\alpha \in Y$ with $X \cap \alpha$ stationary in α .

60. Theorem. *Suppose $V = L$, κ is weakly compact and that $X \subset \kappa$ and $Y \subset \text{reg } \kappa$. If X strongly reflects to Y , then X \diamond -reflects to Y .*

Proof. Define D_α by induction on $\alpha \in Y$. For the purpose of the proof also define C_α for each α as follows. Suppose (D_β, C_β) is defined for all $\beta < \alpha$. Let (D, C) be the L -least¹ pair such that

- (1) C is cub subset of α .
- (2) D is a stationary subset of $X \cap \alpha$
- (3) for all $\beta \in Y \cap C$, $D \cap \beta \neq D_\beta$

If there is no such pair then set $D = C = \emptyset$. Then let $D_\alpha = D$ and $C_\alpha = C$. We claim that the sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ is as needed. To show this, let us make a counter assumption: there is a stationary subset Z of X and a cub subset C of κ such that

$$C \cap Y \subset \{\alpha \in Y \mid D_\alpha \neq Z \cap \alpha\}. \quad (\star)$$

Let (Z, C) be the L -least such pair. Let $\lambda > \kappa$ be regular and let M be an elementary submodel of L_λ such that

- (1) $|M| < \kappa$,
- (2) $\alpha = M \cap \kappa \in Y \cap C$,
- (3) $Z \cap \alpha$ is stationary in α ,
- (4) $\{Z, C, X, Y, \kappa\} \subset M$

(2) and (3) are possible by the definition of strong reflection. Let \bar{M} be the Mostowski collapse of M and let $G: M \rightarrow \bar{M}$ be the Mostowski isomorphism. Then $\bar{M} = L_\gamma$ for some $\gamma > \alpha$. Since $\kappa \cap M = \alpha$, we have

¹The least in the canonical definable ordering on L , see [19].

$$G(Z) = Z \cap \alpha, G(C) = C \cap \alpha, G(X) = X \cap \alpha, G(Y) = Y \cap \alpha \text{ and } G(\kappa) = \alpha, \\ (\star\star).$$

Note that by the definability of the canonical ordering of L , the sequence $\langle D_\beta \rangle_{\beta < \kappa}$ is definable. Let $\varphi(x, y, \alpha)$ be the formula which says

“(x, y) is the L -least pair such that x is contained in $X \cap \alpha$, x is stationary in α ,
 y is cub in α and $x \cap \beta \neq D_\beta$ for all $\beta \in y \cap Y \cap \alpha$ ”.

By the assumption,

$$L \models \varphi(Z, C, \kappa), \text{ so } M \models \varphi(Z, C, \kappa) \text{ and } L_\gamma \models \varphi(G(Z), G(C), G(\kappa)).$$

Let us show that this implies $L \models \varphi(G(Z), G(C), G(\kappa))$, i.e. $L \models \varphi(Z \cap \alpha, C \cap \alpha, \alpha)$. This will be a contradiction because then $D_\alpha = Z \cap \alpha$ which contradicts the assumptions (2) and (\star) above.

By the relative absoluteness of being the L -least, the relativised formula with parameters $\varphi^{L_\gamma}(G(Z), G(C), G(\kappa))$ says

“($G(Z), G(C)$) is the L -least pair such that $G(Z)$ is contained in $G(X)$, $G(Z)$ is
(stationary) $^{L_\gamma}$ in $G(\kappa)$, $G(C)$ is cub in $G(\kappa)$ and $G(Z) \cap \beta \neq D_\beta^{L_\gamma}$ for all
 $\beta \in G(C) \cap G(Y) \cap G(\kappa)$ ”.

Written out this is equivalent to

“($Z \cap \alpha, C \cap \alpha$) is the L -least pair such that $Z \cap \alpha$ is contained in $X \cap \alpha$, $Z \cap \alpha$
is (stationary) $^{L_\gamma}$ in α , $C \cap \alpha$ is cub in α and $Z \cap \beta \neq D_\beta^{L_\gamma}$ for all $\beta \in C \cap Y \cap \alpha$ ”.

Note that this is true in L . Since $Z \cap \alpha$ is stationary in α also in L by (3), it remains to show by induction on $\beta \in \alpha \cap Y$ that $Z \cap \alpha \cap D_\beta^{L_\gamma} = D_\beta^L$ and $C_\beta^{L_\gamma} = C_\beta^L$ and we are done. Suppose we have proved this for $\delta \in \beta \cap Y$ and $\beta \in \alpha \cap Y$. Then $(D_\beta^{L_\gamma}, C_\beta^{L_\gamma})$ is

- (a) (the least L -pair) $^{L_\gamma}$ such that
- (b) $(C_\beta$ is a cub subset of $\beta)$ $^{L_\gamma}$,
- (c) $(D_\beta$ is a stationary subset of $\beta)$ $^{L_\gamma}$
- (d) and for all $\delta \in Y \cap \beta$, $(D_\beta \cap \delta \neq D_\delta)$ $^{L_\gamma}$.
- (e) Or there is no such pair and $D_\beta = \emptyset$.

The L -order is absolute as explained above, so (a) is equivalent to (the least L -pair) L . Being a cub subset of α is also absolute for L_γ so (b) is equivalent to

(C_β is a cub subset of α) L . All subsets of β in L are elements of $L_{|\beta|+}$ (see [19]), and since α is regular and $\beta < \alpha \leq \gamma$, we have $\mathcal{P}(\beta) \subset L_\gamma$. Thus

$$(D_\beta \text{ is stationary subset of } \beta)^{L_\gamma} \iff (D_\beta \text{ is stationary subset of } \beta)^L.$$

Finally the statement of (d), $(D_\beta \cap \delta \neq D_\delta)^{L_\gamma}$ is equivalent to $D_\beta \cap \delta \neq D_\delta^{L_\gamma}$ as it is defining D_β , but by the induction hypothesis $D_\delta^{L_\gamma} = D_\delta^L$, so we are done. For (e), the fact that

$$\mathcal{P}(\beta) \subset L_{|\beta|+} \subset L_\alpha \subset L_\gamma$$

as above implies that if there is no such pair in L_γ , then there is no such pair in L . \square

Proof of Theorem 56. In the case (A) we will show that S_λ^κ strongly reflects to $\text{reg}(\kappa)$ in L which suffices by Theorems 59 and 60. For (B) we will assume that κ is a weakly compact cardinal in L and then collapse it to ω_2 to get a \diamond -sequence which witnesses that $S_\omega^{\omega_2} \diamond$ -reflects to $S_{\omega_1}^{\omega_2}$ which is sufficient by Theorem 59. In the following we assume: $V = L$ and κ is weakly compact.

(A): Let us use Π_1^1 -reflection. Let $X \subset S_\lambda^\kappa$. We want to show that the set

$$\{\lambda \in \text{reg}(\kappa) \mid X \cap \lambda \text{ is stationary in } \lambda\}$$

is stationary. Let $C \subset \kappa$ be cub. The sentence

$$“(X \text{ is stationary in } \kappa) \wedge (C \text{ is cub in } \kappa) \wedge (\kappa \text{ is regular})”$$

is a Π_1^1 -property of (V_κ, X, C) . By Π_1^1 -reflection we get $\delta < \kappa$ such that $(V_\delta, X \cap \delta, C \cap \delta)$ satisfies it. But then δ is regular, $X \cap \delta$ is stationary and δ belongs to C .

(B): Let κ be weakly compact and let us Levy-collapse κ to ω_2 with the following forcing:

$$\mathbb{P} = \{f: \text{reg } \kappa \rightarrow \kappa^{<\omega_1} \mid \text{ran}(f(\mu)) \subset \mu, |\{\mu \mid f(\mu) \neq \emptyset\}| \leq \omega\}.$$

Order \mathbb{P} by $f < g$ if and only if $f(\mu) \subset g(\mu)$ for all $\mu \in \text{reg}(\kappa)$. For all μ put $\mathbb{P}_\mu = \{f \in \mathbb{P} \mid \text{sprt } f \subset \mu\}$ and $\mathbb{P}^\mu = \{f \in \mathbb{P} \mid \text{sprt } f \subset \kappa \setminus \mu\}$, where sprt means support, see Section 2.1.2 on page 5.

Claim 1. For all regular μ , $\omega < \mu \leq \kappa$, \mathbb{P}_μ satisfies the following:

- (a) If $\mu > \omega_1$, then \mathbb{P}_μ has the μ -c.c.,
- (b) \mathbb{P}_μ and \mathbb{P}^μ are $< \omega_1$ -closed,
- (c) $\mathbb{P} = \mathbb{P}_\kappa \Vdash \omega_2 = \check{\kappa}$,

- (d) If $\mu < \kappa$, then $\mathbb{P} \Vdash \text{cf}(\check{\mu}) = \omega_1$,
- (e) if $p \in \mathbb{P}$, σ a name and $p \Vdash \text{"}\sigma \text{ is cub in } \omega_2\text{"}$, then there is cub $E \subset \kappa$ such that $p \Vdash \check{E} \subset \sigma$.

Proof. Standard (see for instance [15]). \square

We want to show that in the generic extension $S_{\omega}^{\omega_2}$ \diamond -reflects to $S_{\omega_1}^{\omega_2}$. It is sufficient to show that $S_{\omega}^{\omega_2}$ \diamond -reflects to some stationary $Y \subset S_{\omega_1}^{\omega_2}$ by letting $D_\alpha = \alpha$ for $\alpha \notin Y$. In our case $Y = \{\mu \in V[G] \mid (\mu \in \text{reg}(\kappa))^V\}$. By (d) of Claim 1, $Y \subset S_{\omega_1}^{\omega_2}$, $(\text{reg}(\kappa))^V$ is stationary in V (for instance by Π_1^1 -reflection) and by (e) it remains stationary in $V[G]$.

It is easy to see that $\mathbb{P} \cong \mathbb{P}_\mu \times \mathbb{P}^\mu$. Let G be a \mathbb{P} -generic over (the ground model) V . Define

$$G_\mu = G \cap \mathbb{P}_\mu.$$

and

$$G^\mu = G \cap \mathbb{P}^\mu.$$

Then G_μ is \mathbb{P}_μ -generic over V .

Also G^μ is \mathbb{P}^μ -generic over $V[G_\mu]$ and $V[G] = V[G_\mu][G^\mu]$.

Let

$$E = \{p \in \mathbb{P} \mid (p > q) \wedge (p_\mu \Vdash p^\mu \in \dot{D})\}.$$

Then E is dense above q : If $p > q$ is arbitrary element of \mathbb{P} , then $q \Vdash \exists p' > \check{p}^\mu (p' \in \dot{D})$ by $(\#)$. Thus there exists $q' > q$ with $q' > p_\mu$, $q' \in \mathbb{P}_\mu$ and $p' > p$, $p' \in \mathbb{P}^\mu$ such that $q' \Vdash p' \in \dot{D}$ and so $(q' \restriction \mu) \cup (p' \restriction (\kappa \setminus \mu))$ is above p and in E . So there is $p \in G \cap E$. But then $p_\mu \in G_\mu$ and $p^\mu \in G^\mu$ and $p_\mu \Vdash p^\mu \in \dot{D}$, so $G^\mu \cap D \neq \emptyset$. Since D was arbitrary, this shows that G^μ is \mathbb{P}^μ -generic over $V[G_\mu]$. Clearly $V[G]$ contains both G_μ and G^μ . On the other hand, $G = G_\mu \cup G^\mu$, so $G \in V[G_\mu][G^\mu]$. By minimality of forcing extensions, we get $V[G] = V[G_\mu][G^\mu]$.

For each $\mu \in \text{reg}(\kappa) \setminus \{\omega, \omega_1\}$ let

$$k_\mu: \mu^+ \rightarrow \{\sigma \mid \sigma \text{ is a nice } \mathbb{P}_\mu \text{ name for a subset of } \mu\}$$

be a bijection. A nice \mathbb{P}_μ name for a subset of $\check{\mu}$ is of the form

$$\bigcup \{\{\check{\alpha}\} \times A_\alpha \mid \alpha \in B\},$$

where $B \subset \check{\mu}$ and for each $\alpha \in B$, A_α is an antichain in \mathbb{P}_μ . By (a) there are no antichains of length μ in \mathbb{P}_μ and $|\mathbb{P}_\mu| = \mu$, so there are at most $\mu^{<\mu} = \mu$ antichains and there are μ^+ subsets $B \subset \mu$, so there indeed exists such a bijection k_μ (these

cardinality facts hold because $V = L$ and μ is regular). Note that if σ is a nice \mathbb{P}_μ -name for a subset of $\check{\mu}$, then $\sigma \subset V_\mu$.

Let us define

$$D_\mu = \begin{cases} \left[k_\mu \left([(\cup G)(\mu^+)](0) \right) \right]_G & \text{if it is stationary} \\ \mu & \text{otherwise.} \end{cases}$$

Now D_μ is defined for all $\mu \in Y$, recall $Y = \{\mu \in V[G] \mid (\mu \in \text{reg } \kappa)^V\}$. We claim that $\langle D_\mu \rangle_{\mu \in Y}$ is the needed \diamond -sequence. Suppose it is not. Then there is a stationary set $S \subset S_\omega^{\omega_2}$ and a cub $C \subset \omega_2$ such that for all $\alpha \in C \cap Y$, $D_\alpha \neq S \cap \alpha$. By (e) there is a cub set $C_0 \subset C$ such that $C_0 \in V$. Let \dot{S} be a nice name for S and p' such that p' forces that \dot{S} is stationary. Let us show that

$$H = \{q \geq p' \mid q \Vdash D_\mu = \dot{S} \cap \check{\mu} \text{ for some } \mu \in C_0\}$$

is dense above p' which is obviously a contradiction. For that purpose let $p > p'$ be arbitrary and let us show that there is $q > p$ in H . Let us now use Π_1^1 -reflection. First let us redefine \mathbb{P} . Let $\mathbb{P}^* = \{q \mid \exists r \in \mathbb{P}(r \restriction \text{sprt } r = q)\}$. Clearly $\mathbb{P}^* \cong \mathbb{P}$ but the advantage is that $\mathbb{P}^* \subset V_\kappa$ and $\mathbb{P}_\mu^* = \mathbb{P}^* \cap V_\mu$ where \mathbb{P}_μ^* is defined as \mathbb{P}_μ . One easily verifies that all the above things (concerning \mathbb{P}_μ , \mathbb{P}^μ etc.) translate between \mathbb{P} and \mathbb{P}^* . From now on denote \mathbb{P}^* by \mathbb{P} . Let

$$R = (\mathbb{P} \times \{0\}) \cup (\dot{S} \times \{1\}) \cup (C_0 \times \{2\}) \cup (\{p\} \times \{3\}).$$

Then $(V_\kappa, R) \models \forall A \varphi$, where φ says: “(if A is closed unbounded and $r > p$ arbitrary, then there exist $q > r$ and α such that $\alpha \in A$ and $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$).” So basically $\forall A \varphi$ says “ $p \Vdash (\dot{S} \text{ is stationary})$ ”. It follows from (e) that it is enough to quantify over cub sets in V . Let us explain why such a formula can be written for (V_κ, R) . The sets (classes from the viewpoint of V_κ) \mathbb{P} , \dot{S} and C_0 are coded into R , so we can use them as parameters. That $r > p$ and $q > r$ and A is closed and unbounded is expressible in first-order as well as $\alpha \in A$. How do we express $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$? The definition of $\check{\alpha}$ is recursive in α :

$$\check{\alpha} = \{(\check{\beta}, 1_{\mathbb{P}}) \mid \beta < \alpha\}$$

and is absolute for V_κ . Then $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$ is equivalent to saying that for each $q' > q$ there exists $q'' > q'$ with $(\check{\alpha}, q'') \in \dot{S}$ and this is expressible in first-order (as we have taken R as a parameter).

By Π_1^1 -reflection there is $\mu \in C_0$ such that $p \in \mathbb{P}_\mu$ and $(V_\mu, R) \models \forall A \varphi$. Note that we may require that μ is regular, i.e. $(\check{\mu}_G \in Y)^{V[G]}$ and such that $\alpha \in S \cap \mu$ implies $(\check{\alpha}, \check{p}) \in \dot{S}$ for some $p \in \mathbb{P}_\mu$. Let $\dot{S}_\mu = \dot{S} \cap V_\mu$.

Thus $p \Vdash_{\mathbb{P}_\mu} \text{“}\dot{S}_\mu \text{ is stationary”}$. Define q as follows: $\text{dom } q = \text{dom } p \cup \{\mu^+\}$, $q \restriction \mu = p \restriction \mu$ and $q(\mu^+) = f$, $\text{dom } f = \{0\}$ and $f(0) = k_\mu^{-1}(\dot{S}_\mu)$. Then $q \Vdash_{\mathbb{P}} \dot{S}_\mu = D_\mu$ provided that $q \Vdash_{\mathbb{P}} \text{“}\dot{S}_\mu \text{ is stationary”}$. The latter holds since \mathbb{P}^μ is $< \omega_1$ -closed, and does not kill stationarity of $(\dot{S}_\mu)_{G_\mu}$ so $(\dot{S}_\mu)_{G_\mu}$ is stationary in $V[G]$ and by the assumption on μ , $(\dot{S}_\mu)_{G_\mu} = (\dot{S}_\mu)_G$. Finally, it remains to show that in $V[G]$, $(\dot{S}_\mu)_G = S \cap \mu$. But this again follows from the definition of μ .

Instead of collapsing κ to ω_2 , we could do the same for λ^{++} for any regular $\lambda < \kappa$ and obtain a model in which $E_{S_\lambda^{++}} \leq_c E_{S_{\lambda^+}^{++}}$. \square

Open Problem. Is it consistent that $S_{\omega_1}^{\omega_2}$ Borel reduces to $S_{\omega_2}^{\omega_2}$?

4.4.3. E_0 and $E_{S_\lambda^\kappa}$. In the Section 4.4.2 above, Theorem 57, we showed that the equivalence relations of the form $E_{S_\lambda^\kappa}$ can form an antichain with respect to \leq_B . We will show that under mild set theoretical assumptions, all of them are strictly above

$$E_0 = \{(\eta, \xi) \mid \eta^{-1}\{1\} \triangle \xi^{-1}\{1\} \text{ is bounded}\}.$$

61. Theorem. *Let κ be regular and $S \subset \kappa$ stationary and suppose that $\diamond_\kappa(S)$ holds (i.e., \diamond_κ holds on the stationary set S). Then E_0 is Borel reducible to E_S .*

Proof. The proof uses similar ideas than the proof of Theorem 59. Suppose that the $\diamond_\kappa(S)$ holds and let $\langle D_\alpha \rangle_{\alpha \in S}$ be the $\diamond_\kappa(S)$ -sequence. Define the reduction $f: 2^\kappa \rightarrow 2^\kappa$ by

$$f(X) = \{\alpha \in S \mid D_\alpha \text{ and } X \cap \alpha \text{ agree on a final segment of } \alpha\}.$$

If X, Y are E_0 -equivalent, then $f(X), f(Y)$ are E_S -equivalent, because they are in fact even E_0 -equivalent as is easy to check. If X, Y are not E_0 -equivalent, then there is a club C of α where X, Y differ cofinally in α ; it follows that $f(X), f(Y)$ differ on a stationary subset of S , namely the elements α of $C \cap S$ where D_α equals $X \cap \alpha$. \square

62. Corollary. *Suppose $\kappa = \lambda^+ = 2^\lambda$. Then E_0 is Borel reducible to E_S where $S \subset \kappa \setminus S_{\text{cf}(\lambda)}^\kappa$ is stationary.*

Proof. Gregory proved in [3] that if $2^\mu = \mu^+ = \kappa$, μ is regular and $\lambda < \mu$, then $\diamond_\kappa(S_\lambda^\kappa)$ holds. Shelah extended this result in [29] and proved that if $\kappa = \lambda^+ = 2^\lambda$ and $S \subset \kappa \setminus S_{\text{cf}(\lambda)}^\kappa$, then $\diamond_\kappa(S)$ holds. Now apply Theorem 61. \square

63. Corollary (GCH). *Let us assume that κ is a successor cardinal. Then in a cofinality and GCH preserving forcing extension, there is an embedding*

$$f: \langle \mathcal{P}(\kappa), \subset \rangle \rightarrow \langle \mathcal{E}^{\Sigma_1^1}, \leq_B \rangle,$$

where $\mathcal{E}^{\Sigma_1^1}$ is the set of Σ_1^1 -equivalence relations (see Theorem 53) such that for all $A \in \mathcal{P}(\kappa)$, E_0 is strictly below $f(A)$. If κ is not the successor of an ω -cofinal cardinal, we may replace Σ_1^1 above by Borel^* .

Proof. Suppose first that κ is not the successor of an ω -cofinal cardinal. By Theorem 53 there is a GCH and cofinality-preserving forcing extension such that there is an embedding

$$f: \langle \mathcal{P}(\kappa), \subset \rangle \rightarrow \langle \mathcal{E}^{\text{Borel}^*}, \leq_B \rangle.$$

From the proof of Theorem 53 one sees that $f(A)$ is of the form E_S where $S \subset S_\omega^\kappa$. Now E_0 is reducible to such relations by Corollary 62, as GCH continues to hold in the extension.

So it suffices to show that $E_S \not\leq_B E_0$ for stationary $S \subset S_\omega^\kappa$. By the same argument as in Corollary 50, E_S is not Borel and by Theorem 34 E_0 is Borel, so by Fact 6 $E_{S_\lambda^\kappa}$ is not reducible to E_0 .

Suppose κ is the successor of an ω -cofinal ordinal and $\kappa > \omega_1$. Then, in the proof of Theorem 53 replace μ by ω_1 and get the same result as above but for relations of the form E_S where $S \subset S_{\omega_1}^\kappa$.

The remaining case is $\kappa = \omega_1$. Let $\{S_\alpha \mid \alpha < \omega_1\}$ be a set of pairwise disjoint stationary subsets of ω_1 . Let \mathbb{P} be the forcing given by the proof of Theorem 53 such that in the \mathbb{P} -generic extension the function $f: \langle \mathcal{P}(\omega_1), \subset \rangle \rightarrow \langle \mathcal{E}^{\text{Borel}^*}, \leq_B \rangle$ given by $f(A) = E_{\bigcup_{\alpha \in A} S_\alpha}$ is an embedding. This forcing preserves stationary sets, so as in the proof of clause (4) of Theorem 49, we can first force a \diamond -sequence which guesses each subset of $\bigcup_{\alpha < \omega_1} S_\alpha$ on a set S such that $S \cap S_\alpha$ is stationary for all α . Then by Corollary 62 E_0 is reducible to $E_{\bigcup_{\alpha \in A} S_\alpha}$ for all $A \subset \kappa$. \square

5. COMPLEXITY OF ISOMORPHISM RELATIONS

Let T be a countable complete theory. Let us turn to the question discussed in Section 1: “How is the set theoretic complexity of \cong_T related to the stability theoretic properties of T ?”. The following theorems give some answers. As pointed out in Section 1, the assumption that κ is uncountable is crucial in the following theorems. For instance the theory of dense linear orderings without end points is unstable, but \cong_T is an open set in case $\kappa = \omega$, while we show

below that for unstable theories T the set \cong_T cannot be even Δ_1^1 when $\kappa > \omega$. Another example introduced by Martin Koerwien in his Ph.D. thesis and in [18] shows that there are classifiable shallow theories whose isomorphism is not Borel when $\kappa = \omega$, although we prove below that the isomorphism of such theories is always Borel, when $\kappa > \omega$. This justifies in particular the motivation for studying the space κ^κ for model theoretic purposes: the set theoretic complexity of \cong_T positively correlates with the model theoretic complexity of T .

The following stability theoretical notions will be used: stable, superstable, DOP, OTOP, shallow, $\lambda(T)$ and $\kappa(T)$. Classifiable means superstable with no DOP nor OTOP and $\lambda(T)$ is the least cardinal in which T is stable.

The main theme in this section is exposed in the following two theorems:

64. Theorem ($\kappa^{<\kappa} = \kappa$). *Assume that κ is not weakly inaccessible. A theory T is classifiable and shallow if and only if its isomorphism relation on structures of size κ is Borel.*

65. Theorem ($\kappa^{<\kappa} = \kappa$). *Assume that for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and $\kappa > \omega_1$. Then in L and in the forcing extension after adding κ^+ Cohen subsets of κ we have: for any theory T , T is classifiable if and only if \cong_T is Δ_1^1 .*

The two theorems above are proved in many subtheorems below. Our results are stronger than those given by 64 and 65 (for instance the cardinality assumption $\kappa > \omega_1$ is needed only in the case where T is superstable with DOP and the stable unsuperstable case is the only one for which Theorem 65 cannot be proved in ZFC). Theorem 64 follows from Theorems 69, 70. Theorem 65 follows from Theorems 71, 72, 73 and items (2) and (3) of Theorem 49.

5.1. Preliminary Results. The following Theorems 66 and 68 will serve as bridges between the set theoretic complexity and the model theoretic complexity of an isomorphism relation.

66. Theorem ($\kappa^{<\kappa} = \kappa$). *For a theory T , the set \cong_T is Borel if and only if the following holds: there exists a $\kappa^+\omega$ -tree t such that for all models \mathcal{A} and \mathcal{B} of T , $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$.*

Proof. Recall that we assume $\text{dom } \mathcal{A} = \kappa$ for all models in the discourse. First suppose that there exists a $\kappa^+\omega$ -tree t such that for all models \mathcal{A} and \mathcal{B} of T , $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Let us show that there exists a $\kappa^+\omega$ -tree u which constitutes a Borel code for \cong_T (see Remark 16 on page 15).

Let u be the tree of sequences of the form

$$\langle (p_0, A_0), f_0, (p_1, A_1), f_1, \dots, (p_n, A_n), f_n \rangle$$

such that for all $i \leq n$

- (1) (p_i, A_i) is a move of player **I** in EF_t^κ , i.e. $p_i \in t$ and $A_i \subset \kappa$ with $|A_i| < \kappa$,
- (2) f_i is a move of player **II** in EF_t^κ , i.e. it is a partial function $\kappa \rightarrow \kappa$ with $|\text{dom } f_i|, |\text{ran } f_i| < \kappa$ and $A_i \subset \text{dom } f_i \cap \text{ran } f_i$
- (3) $\langle (p_0, A_0), f_0, (p_1, A_1), f_1, \dots, (p_n, A_n), f_n \rangle$ is a valid position of the game, i.e. $(p_i)_{i \leq n}$ is an initial segment of a branch in t and $A_i \subset A_j$ and $f_i \subset f_j$ whenever $i < j \leq n$.

Order u by end extension. The tree u is a $\kappa^+\omega$ -tree (because t is and by (3)).

Let us now define the function

$$h: \{\text{branches of } u\} \rightarrow \{\text{basic open sets of } (\kappa^\kappa)^2\}.$$

Let $b \subset u$ be a branch,

$$b = \{\emptyset, \langle (p_0, A_0) \rangle, \langle (p_0, A_0), f_0 \rangle, \dots, \langle (p_0, A_0), f_0, \dots, (p_k, A_k), f_k \rangle\}.$$

It corresponds to a unique EF-game between some two structures with domains κ . In this game the players have chosen some set $A_k = \bigcup_{i \leq k} A_i \subset \kappa$ and some partial function $f_k = \bigcup_{i \leq k} f_i: \kappa \rightarrow \kappa$. Let $h(b)$ be the set of all pairs $(\eta, \xi) \in (\kappa^\kappa)^2$ such that $f_k: \mathcal{A}_\eta \upharpoonright A_k \cong \mathcal{A}_\xi \upharpoonright A_k$ is a partial isomorphism. This is clearly an open set:

$$(\eta, \xi) \in h(b) \Rightarrow N_{\eta \upharpoonright ((\sup A_k)+1)} \times N_{\xi \upharpoonright ((\sup A_k)+1)} \subset h(b).$$

Finally we claim that $\mathcal{A}_\eta \cong \mathcal{A}_\xi \iff \mathbf{II} \upharpoonright G(u, h, (\eta, \xi))$. Here G is the game as in Definition 15 of Borel* sets, page 14 but played on the product $\kappa^\kappa \times \kappa^\kappa$. Assume $\mathcal{A}_\eta \cong \mathcal{A}_\xi$. Then $\mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)$. Let v denote the winning strategy. In the game $G(u, h, (\eta, \xi))$, let us define a winning strategy for player **II** as follows. By definition, at a particular move, say n , **I** chooses a sequence

$$\langle (p_0, A_0), f_0, \dots, (p_n, A_n) \rangle.$$

Next **II** extends it according to v to

$$\langle (p_0, A_0), f_0, \dots, (p_n, A_n), f_n \rangle,$$

where $f_n = v((p_0, A_0), \dots, (p_n, A_n))$. Since v was a winning strategy, it is clear that $f_k = \bigcup_{i < k} f_i$ is going to be a isomorphism between $\mathcal{A}_\eta \upharpoonright A_k$ and $\mathcal{A}_\xi \upharpoonright A_k$, so $(\eta, \xi) \in h(b)$.

Assume that $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$. Then by the assumption there is no winning strategy of **II**, so player **I** can play in such a way that $f_\kappa = \bigcup_{i \leq \kappa} f_i$ is not an isomorphism between $\mathcal{A}_\eta \upharpoonright \bigcup A_i$ and $\mathcal{A}_\xi \upharpoonright \bigcup A_i$, so (η, ξ) is not in $h(b)$. This completes the proof of the direction “ \Leftarrow ”

Let us prove “ \Rightarrow ”. Suppose \cong_T is Borel and let us show that there is a tree as in the statement of the theorem. We want to use Theorem 24 and formalize the statement “ \cong_T is definable in $L_{\kappa+\kappa}$ ” by considering the space consisting of pairs of models.

Denote the vocabulary of \mathcal{A} and \mathcal{B} as usual by L . Let P be a unary relation symbol not in L . We will now discuss two distinct vocabularies, L and $L \cup \{P\}$ at the same time, so we have to introduce two distinct codings. Fix an $\eta \in 2^\kappa$. Let \mathcal{A}_η denote the L -structure as defined in Definition 12 of our usual coding. Let $\rho: \kappa \cup \kappa^{<\omega} \rightarrow \kappa$ be a bijection and define \mathcal{A}^η to be the model with $\text{dom } \mathcal{A}^\eta = \kappa$ and if $a \in \text{dom } \mathcal{A}^\eta$, then $\mathcal{A}^\eta \models P(a) \iff \eta(\rho(a)) = 1$ such that if $(a_1, \dots, a_n) \in (\text{dom } \mathcal{A}^\eta)^n$, then $\mathcal{A}^\eta \models P_n(a_1, \dots, a_n) \iff \eta(\rho(a_1, \dots, a_n)) = 1$. Note that we are making a distinction here between κ and $\kappa^{\{0\}}$.

Claim 1. The set $W = \{\eta \in 2^\kappa \mid \kappa = |P^{\mathcal{A}^\eta}| = |\kappa \setminus P^{\mathcal{A}^\eta}|\}$ is Borel.

Proof of Claim 1. Let us show that the complement is Borel. By symmetry it is sufficient to show that

$$B = \{\eta \mid \kappa > |P^{\mathcal{A}^\eta}|\}$$

is Borel. Let $I \subset \kappa$ be a subset of size $< \kappa$. For $\beta \notin I$ define $U(I, \beta)$ to be the set

$$U(I, \beta) = \{\eta \mid \eta(\rho(\beta)) = 0\}.$$

Clearly $U(I, \beta)$ is open for all I, β . Now

$$B = \bigcup_{I \in [\kappa]^{<\kappa}} \bigcap_{\beta \notin I} U(I, \beta).$$

By the assumption $\kappa^{<\kappa} = \kappa$, this is Borel (in fact a union of closed sets). $\square_{\text{Claim 1}}$

Define a mapping $h: W \rightarrow (2^\kappa)^2$ as follows. Suppose $\xi \in W$. Let

$$r_1: \kappa \rightarrow P^{\mathcal{A}^\xi}$$

and

$$r_2: \kappa \rightarrow \kappa \setminus P^{\mathcal{A}^\xi}$$

be the order preserving bijections (note $P^{\mathcal{A}^\eta} \subset \kappa = \text{dom } \mathcal{A}^\eta$).

Let η_1 be such that r_1 is an isomorphism

$$\mathcal{A}_{\eta_1} \rightarrow (\mathcal{A}^\xi \cap P^{\mathcal{A}^\xi}) \upharpoonright L$$

and η_2 such that r_2 is an isomorphism

$$\mathcal{A}_{\eta_2} \rightarrow (\mathcal{A}^\xi \setminus P^{\mathcal{A}^\xi}) \upharpoonright L.$$

Clearly η_1 and η_2 are unique, so we can define $h(\xi) = (\eta_1, \eta_2)$.

Claim 2. h is continuous.

Proof of Claim 2. Let $U = N_p \times N_q$ be a basic open set of $(2^\kappa)^2$, $p, q \in 2^{<\kappa}$ and let $\xi \in h^{-1}[U]$. Let $P^{\mathcal{A}^\xi} = \{\beta_i \mid i < \kappa\}$ be an enumeration such that $\beta_i < \beta_j \iff i < j$ and similarly $\kappa \setminus P^{\mathcal{A}^\xi} = \{\gamma_i \mid i < \kappa\}$. Let $\alpha = \max\{\beta_{\text{dom } p}, \gamma_{\text{dom } q}\} + 1$. Then $N_{\xi \upharpoonright \alpha} \subset h^{-1}[U]$. Thus arbitrary ξ in $h^{-1}[U]$ have an open neighbourhood in $h^{-1}[U]$, so it is open. \square Claim 2

Recall our assumption that $E = \{(\eta, \xi) \in 2^\kappa \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$ is Borel. Since h is continuous and in particular Borel, this implies that

$$E' = \{\eta \mid \mathcal{A}_{h_1(\eta)} \cong \mathcal{A}_{h_2(\eta)}\} = h^{-1}E$$

is Borel in W . Because W is itself Borel, E' is Borel in 2^κ . Additionally, E' is closed under permutations: if \mathcal{A}^η is isomorphic to \mathcal{A}^ξ , then $\mathcal{A}^\eta \cap P^{\mathcal{A}^\eta}$ is isomorphic to $\mathcal{A}^\xi \cap P^{\mathcal{A}^\xi}$ and $\mathcal{A}^\eta \setminus P^{\mathcal{A}^\eta}$ is isomorphic to $\mathcal{A}^\xi \setminus P^{\mathcal{A}^\xi}$, so if $\mathcal{A}^\eta \in E'$, then also $\mathcal{A}^\xi \in E'$ (and note that since $\eta \in W$, also $\xi \in W$). By Theorem 24, there is a sentence θ of $L_{\kappa+\kappa}$ over $L \cup \{P\}$ that defines E' . Thus by Theorem 9 and Remark 11 there is a $\kappa^+\omega$ -tree t such that

$$\text{if } \eta \in E' \text{ and } \xi \notin E', \text{ then } \mathbf{II} \not\upharpoonright \text{EF}_t^\kappa(\mathcal{A}^\eta, \mathcal{A}^\xi). \quad \odot$$

We claim that t is as needed, i.e. for all models \mathcal{A}, \mathcal{B} of T

$$\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B}).$$

Suppose not. Then there are models $\mathcal{A} \not\cong \mathcal{B}$ such that $\mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Let η and ξ be such that $\mathcal{A}_{h_1(\eta)} = \mathcal{A}_{h_2(\eta)} = \mathcal{A}_{h_1(\xi)} = \mathcal{A}$ and $\mathcal{A}_{h_2(\xi)} = \mathcal{B}$. Clearly $\eta \in E'$, but $\xi \notin E'$, so by \odot there is no winning strategy of \mathbf{II} in $\text{EF}_t^\kappa(\mathcal{A}^\eta, \mathcal{A}^\xi)$ which is clearly a contradiction, because \mathbf{II} can apply her winning strategies in $\text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ and $\text{EF}_t^\kappa(\mathcal{A}, \mathcal{A})$ to win in $\text{EF}_t^\kappa(\mathcal{A}^\eta, \mathcal{A}^\xi)$. \square Theorem 66

We will use the following lemma from [22]:

67. **Lemma.** If $t \subset (\kappa^{<\kappa})^2$ is a tree and $\xi \in \kappa^\kappa$, denote

$$t(\xi) = \{p \in \kappa^{<\kappa} \mid (p, \xi \upharpoonright \text{dom } p) \in t\}.$$

Similarly if $t \in (\kappa^{<\kappa})^3$, then

$$t(\eta, \xi) = \{p \in \kappa^{<\kappa} \mid (p, \eta \upharpoonright \text{dom } p, \xi \upharpoonright \text{dom } p) \in t\}.$$

Assume that Z is Σ_1^1 . Then Z is Δ_1^1 if and only if for every tree $t \subset (\kappa^{<\kappa})^2$ such that

$$t(\xi) \text{ has a } \kappa\text{-branch} \iff \xi \in Z$$

there exists a $\kappa^+\kappa$ -tree t' such that $\xi \in Z \iff t(\xi) \not\leq t'$. (Recall that $t \leq t'$ when there exists a strictly order preserving map $t \rightarrow t'$)

68. **Theorem.** Let T be a theory and assume that for every $\kappa^+\kappa$ -tree t there exist $(\eta, \xi) \in (2^\kappa)^2$ such that $\mathcal{A}_\eta, \mathcal{A}_\xi \models T$, $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$ but $\mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)$. Then \cong_T is not Δ_1^1 .

Proof. Let us abbreviate some statements:

$A(t)$: $t \subset (\kappa^{<\kappa})^3$ is a tree and for all $(\eta, \xi) \in (\kappa^\kappa)^2$,

$$(\eta, \xi) \in \cong_T \iff t(\eta, \xi) \text{ contains a } \kappa\text{-branch}.$$

$B(t, t')$: $t \subset (\kappa^{<\kappa})^3$ is a $\kappa^+\kappa$ -tree and for all $(\eta, \xi) \in \kappa^\kappa$,

$$(\eta, \xi) \in \cong_T \iff t(\eta, \xi) \not\leq t'.$$

Now Lemma 67 implies that if \cong_T is Δ_1^1 , then $\forall t[A(t) \rightarrow \exists t'B(t, t')]$. We will show that $\exists t[A(t) \wedge \forall t' \neg B(t, t')]$, which by Lemma 67 suffices to prove the theorem. Let us define t . In the following, ν_α , η_α and ξ_α stand respectively for $\nu \upharpoonright \alpha$, $\eta \upharpoonright \alpha$ and $\xi \upharpoonright \alpha$.

$$t = \{(\nu_\alpha, \eta_\alpha, \xi_\alpha) \mid \alpha < \kappa \text{ and } \nu \text{ codes an isomorphism between } \mathcal{A}_\eta \text{ and } \mathcal{A}_\xi\}.$$

Using Theorem 13 it is easy to see that t satisfies $A(t)$. Assume now that t' is an arbitrary $\kappa^+\kappa$ -tree. We will show that $B(t, t')$ does not hold. For that purpose let $u = \omega \times t'$ be the tree defined by the set $\{(n, s) \mid n \in \omega, s \in t'\}$ and the ordering

$$(n_0, s_0) <_u (n_1, s_1) \iff (s_0 <_{t'} s_1 \vee (s_0 = s_1 \wedge n_0 <_\omega n_1)). \quad (1)$$

This tree u is still a $\kappa^+\kappa$ -tree, so by the assumption of the theorem there is a pair (ξ_1, ξ_2) such that \mathcal{A}_{ξ_1} and \mathcal{A}_{ξ_2} are non-isomorphic, but $\mathbf{II} \upharpoonright \text{EF}_u^\kappa(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$.

It is now sufficient to show that $t(\xi_1, \xi_2) \not\leq t'$.

Claim 1. There is no order preserving function

$$\sigma t' \rightarrow t',$$

where $\sigma t'$ is defined in Definition 31.

Proof of Claim 1. Assume $g: \sigma t' \rightarrow t'$, is order preserving. Define $x_0 = g(\emptyset)$ and

$$x_\alpha = g(\{y \in t' \mid \exists \beta < \alpha (y \leq x_\beta)\}) \text{ for } 0 < \alpha < \kappa$$

Then $(x_\alpha)_{\alpha < \kappa}$ contradicts the assumption that t' is a $\kappa^+ \kappa$ -tree. $\square_{\text{Claim 1}}$

Claim 2. There is an order preserving function

$$\sigma t' \rightarrow t(\xi_1, \xi_2).$$

Proof of Claim 2. The idea is that players **I** and **II** play an EF-game for each branch of the tree t' and **II** uses her winning strategy in $\text{EF}_u^\kappa(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$ to embed that branch into the tree of partial isomorphisms. A problem is that the winning strategy gives arbitrary partial isomorphisms while we are interested in those which are coded by functions defined on page 13. Now the tree u of (1) above becomes useful.

Let σ be a winning strategy of player **II** in $\text{EF}_u^\kappa(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$. Let us define $g: \sigma t' \rightarrow t(\xi_1, \xi_2)$ recursively. Recall the function π from Definition 12 and define

$$C = \{\alpha \mid \pi[\alpha^{<\omega}] = \alpha\}.$$

Clearly C is cub. If $s \subset t'$ is an element of $\sigma t'$, then we assume that g is defined for all $s' <_{\sigma t'} s$ and that EF_u^κ is played up to $(0, \sup s) \in u$. If s does not contain its supremum, then put $g(s) = \bigcup_{s' < s} g(s')$. Otherwise let them continue playing the game for ω more moves; at the n^{th} of these moves player **I** picks $(n, \sup s)$ from u and a $\beta < \kappa$ where β is an element of C above

$$\max\{\text{ran } f_{n-1}, \text{dom } f_{n-1}\}$$

where f_{n-1} is the previous move by **II**. (If $n = 0$, it does not matter what **I** does.) In that way the function $f = \bigcup_{n < \omega} f_n$ is a partial isomorphism such that $\text{dom } f = \text{ran } f = \alpha$ for some ordinal α . It is straightforward to check that such an f is coded by some $\nu_\alpha: \alpha \rightarrow \kappa$. It is an isomorphism between $\mathcal{A}_{\xi_1} \cap \alpha$ and $\mathcal{A}_{\xi_2} \cap \alpha$ and since α is in C , there are ξ'_1 and ξ'_2 such that $\xi_1 \restriction \alpha \subset \xi'_1$, $\xi_2 \restriction \alpha \subset \xi'_2$ and there is an isomorphism $\mathcal{A}_{\xi'_1} \cong \mathcal{A}_{\xi'_2}$ coded by some ν such that $\nu_\alpha = \nu \restriction \alpha$.

Thus $\nu_\alpha \in t(\xi_1, \xi_2)$ is suitable for setting $g(s) = \nu_\alpha$. \square Claim 2

\square Theorem 68

5.2. Classifiable. Throughout this section κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa > \omega$.

69. Theorem. *If the theory T is classifiable and shallow, then \cong_T is Borel.*

Proof. If T is classifiable and shallow, then from [25] Theorem XIII.1.5 it follows that the models of T are characterized by the game EF_t^κ up to isomorphism, where t is some $\kappa^+\omega$ -tree (in fact a tree of descending sequences of an ordinal $\alpha < \kappa^+$). Hence by Theorem 66 the isomorphism relation of T is Borel. \square

70. Theorem. *If the theory T is classifiable but not shallow, then \cong_T is not Borel. If κ is not weakly inaccessible and T is not classifiable, then \cong_T is not Borel.*

Proof. If T is classifiable but not shallow, then by [25] XIII.1.8, the $L_{\infty\kappa}$ -Scott heights of models of T of size κ are not bounded by any ordinal $< \kappa^+$ (see Definition 7 on page 11). Because any $\kappa^+\omega$ -tree can be embedded into $t_\alpha = \{\text{decreasing sequences of } \alpha\}$ for some α (see Fact 2.2.1 on page 7), this implies that for any $\kappa^+\omega$ -tree t there exists a pair of models \mathcal{A}, \mathcal{B} such that $\mathcal{A} \not\cong \mathcal{B}$ but $\mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Theorem 66 now implies that the isomorphism relation is not Borel.

If T is not classifiable κ is not weakly inaccessible, then by [26] Theorem 0.2 (Main Conclusion), there are non-isomorphic models of T of size κ which are $L_{\infty\kappa}$ -equivalent, so the same argument as above, using Theorem 66, gives that \cong_T is not Borel. \square

71. Theorem. *If the theory T is classifiable, then \cong_T is Δ_1^1 .*

Proof. Shelah's theorem [25] XIII.1.4 implies that if a theory T is classifiable, then any two models that are $L_{\infty\kappa}$ -equivalent are isomorphic. But $L_{\infty\kappa}$ equivalence is equivalent to EF_ω^κ -equivalence (see Theorem 10 on page 12). So in order to prove the theorem it is sufficient to show that if for any two models \mathcal{A}, \mathcal{B} of the theory T it holds that $\mathbf{II} \upharpoonright \text{EF}_\omega^\kappa(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \cong \mathcal{B}$, then the isomorphism relation is Δ_1^1 . The game EF_ω^κ is a closed game of length ω and so determined. Hence we have $\mathbf{I} \upharpoonright \text{EF}_\omega^\kappa(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \not\cong \mathcal{B}$. By Theorem 6 the set

$$\{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \text{ codes a winning strategy for } \mathbf{I} \upharpoonright \text{EF}_\omega^\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)\}$$

is closed and thus $\{(\eta, \xi) \mid \mathcal{A}_\eta \not\cong \mathcal{A}_\xi\}$ is Σ_1^1 , which further implies that \cong_T is Δ_1^1 by Corollary 14. \square

5.3. Unclassifiable.

5.3.1. *The Unstable, DOP and OTOP Cases.* As before, κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa > \omega$.

72. Theorem. (1) *If T is unstable then \cong_T is not Δ_1^1 .*

(2) *If T is stable with OTOP, then \cong_T is not Δ_1^1 .*

(3) *If T is superstable with DOP and $\kappa > \omega_1$, then \cong_T is not Δ_1^1 .*

(4) *If T is stable with DOP and $\lambda = \text{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geq \omega_1, \kappa > \lambda^+$ and for all $\xi < \kappa, \xi^\lambda < \kappa$, then \cong_T is not Δ_1^1 . (Note that $\kappa(T) \in \{\omega, \omega_1\}$.)*

Proof. For a model \mathcal{A} of size κ of a theory T let us denote by $E(\mathcal{A})$ the following property: for every $\kappa^+\kappa$ -tree t there is a model \mathcal{B} of T of cardinality κ such that $\mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ and $\mathcal{A} \not\cong \mathcal{B}$.

For (3) we need a result by Hyttinen and Tuuri, Theorem 6.2. from [14]:

Fact (Superstable with DOP). *Let T be a superstable theory with DOP and $\kappa^{<\kappa} = \kappa > \omega_1$. Then there exists a model \mathcal{A} of T of cardinality κ with the property $E(\mathcal{A})$.*

For (4) we will need a result by Hyttinen and Shelah from [13]:

Fact (Stable with DOP). *Let T be a stable theory with DOP and $\lambda = \text{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geq \omega_1, \kappa^{<\kappa} = \kappa > \lambda^+$ and for all $\xi < \kappa, \xi^\lambda < \kappa$. Then there is a model \mathcal{A} of T of power κ with the property $E(\mathcal{A})$.*

For (1) a result by Hyttinen and Tuuri Theorem 4.9 from [14]:

Fact (Unstable). *Let T be an unstable theory. Then there exists a model \mathcal{A} of T of cardinality κ with the property $E(\mathcal{A})$.*

And for (2) another result by Hyttinen and Tuuri, Theorem 6.6 in [14]:

Fact (Stable with OTOP). *Suppose T is a stable theory with OTOP. Then there exists a model \mathcal{A} of T of cardinality κ with the property $E(\mathcal{A})$.*

Now (1), (2) and (4) follow immediately from Theorem 68. \square

5.3.2. *Stable Unsuperstable.* We assume $\kappa^{<\kappa} = \kappa > \omega$ in all theorems below.

73. Theorem. *Assume that for all $\lambda < \kappa, \lambda^\omega < \kappa$.*

(1) *If T is stable unsuperstable, then \cong_T is not Borel.*

(2) If κ is as above and T is stable unsuperstable, then \cong_T is not Δ_1^1 in the forcing extension after adding κ^+ Cohen subsets of κ , or if $V = L$.

Proof. By Theorem 87 on page 94 the relation $E_{S_\kappa^\kappa}$ can be reduced to \cong_T . The theorem follows now from Corollary 50 on page 49. \square

On the other hand, stable unsuperstable theories sometimes behave nicely to some extent:

74. Lemma. Assume that T is a theory and t a $\kappa^+\kappa$ -tree such that if \mathcal{A} and \mathcal{B} are models of T , then $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \upharpoonright \text{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Then \cong of T is Borel*.

Proof. Similar to the proof of Theorem 66. \square

75. Theorem. Assume $\kappa \in I[\kappa]$ and $\kappa = \lambda^+$ (“ $\kappa \in I[\kappa]$ ” is known as the Approachability Property and follows from $\lambda^{<\lambda} = \lambda$). Then there exists an unsuperstable theory T whose isomorphism relation is Borel*.

Proof. In [11] and [12] Hyttinen and Shelah show the following (Theorem 1.1 of [12], but the proof is essentially in [11]):

Suppose $T = ((\omega^\omega, E_i)_{i < \omega})$, where $\eta E_i \xi$ if and only if for all $j \leq i$, $\eta(j) = \xi(j)$. If $\kappa \in I[\kappa]$, $\kappa = \lambda^+$ and \mathcal{A} and \mathcal{B} are models of T of cardinality κ , then $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \upharpoonright \text{EF}_{\lambda \cdot \omega + 2}^\kappa(\mathcal{A}, \mathcal{B})$, where $+$ and \cdot denote the ordinal sum and product, i.e. $\lambda \cdot \omega + 2$ is just an ordinal.

So taking the tree t to be $\lambda \cdot \omega + 2$ the claim follows from Lemma 74. \square

Open Problem. We proved that the isomorphism relation of a theory T is Borel if and only if T is classifiable and shallow. Is there a connection between the depth of a shallow theory and the Borel degree of its isomorphism relation? Is one monotone in the other?

Open Problem. Can it be proved in ZFC that if T is stable unsuperstable then \cong_T is not Δ_1^1 ?

6. REDUCTIONS

Recall that in Chapter 5 we obtained a provable characterization of theories which are both classifiable and shallow in terms of the definability of their isomorphism relations. Without the shallowness condition we obtained only a consistency result. In this chapter we improve this to a provable characterization by analyzing isomorphism relations in terms of Borel reducibility.

Recall the definition of a reduction, Section 2.1.4 and recall that if $X \subset \kappa$ be a stationary subset, we denote by E_X the equivalence relation defined by

$$\forall \eta, \xi \in 2^\kappa (\eta E_X \xi \iff (\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}) \cap X \text{ is non-stationary}),$$

and by S_λ^κ we mean the ordinals of cofinality λ that are less than κ .

The equivalence relations E_X are Σ_1^1 ($AE_X B$ if and only if *there exists* a cub subset of $\kappa \setminus (X \cap (A \triangle B))$).

Simple conclusions can readily be made from the following observation that roughly speaking, the set theoretic complexity of a relation does not decrease under reductions:

Fact. *If E_1 is a Borel (or Δ_1^1) equivalence relation and E_0 is an equivalence relation with $E_0 \leqslant_B E_1$, then E_0 is Borel (respectively Δ_1^1 if E_1 is Δ_1^1).* \square

The main theorem of this chapter is:

76. Theorem. *Suppose $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Let T be a first-order theory. Then T is classifiable if and only if for all regular $\mu < \kappa$, $E_{S_\mu^\kappa} \not\leqslant_B \cong_T$.*

6.1. Classifiable Theories. The following follows from [25] Theorem XIII.1.4.

77. Theorem ([25]). *If a first-order theory T is classifiable and \mathcal{A} and \mathcal{B} are non-isomorphic models of T of size κ , then $\mathbf{I} \upharpoonright \text{EF}_\omega^\kappa(\mathcal{A}, \mathcal{B})$.* \square

78. Theorem ($\kappa^{<\kappa} = \kappa$). *If a first-order theory T is classifiable, then for all $\lambda < \kappa$*

$$E_{S_\lambda^\kappa} \not\leqslant_B \cong_T.$$

Proof. Let $\text{NS} \in \{E_{S_\lambda^\kappa} \mid \lambda \in \text{reg}(\kappa)\}$.

Suppose $r: 2^\kappa \rightarrow 2^\kappa$ is a Borel function such that

$$\forall \eta, \xi \in 2^\kappa (\mathcal{A}_{r(\eta)} \models T \wedge \mathcal{A}_{r(\xi)} \models T \wedge (\eta \text{ NS } \xi \iff \mathcal{A}_{r(\eta)} \cong \mathcal{A}_{r(\xi)})). \quad (\nabla)$$

By Lemma 33, let D be an intersection of κ -many dense open sets such that $R = r \upharpoonright D$ is continuous. D can be coded into a function $v: \kappa \times \kappa \rightarrow \kappa^{<\kappa}$ such that $D = \bigcap_{i < \kappa} \bigcup_{j < \kappa} N_{v(i,j)}$. Since R is continuous, it can also be coded into a single function $u: \kappa^{<\kappa} \times \kappa^{<\kappa} \rightarrow \{0, 1\}$ such that

$$R(\eta) = \xi \iff (\forall \alpha < \kappa)(\exists \beta < \kappa)[u(\eta \upharpoonright \beta, \xi \upharpoonright \alpha) = 1].$$

(For example define $u(p, q) = 1$ if $D \cap N_p \subset R^{-1}[N_q]$.) Let

$$\varphi(\eta, \xi, u, v) = (\forall \alpha < \kappa)(\exists \beta < \kappa)[u(\eta \upharpoonright \beta, \xi \upharpoonright \alpha) = 1] \wedge (\forall i < \kappa)(\exists j < \kappa)[\eta \in N_{v(i,j)}].$$

It is a formula of set theory with parameters u and v . It is easily seen that φ is absolute for transitive elementary submodels M of $H(\kappa^+)$ containing κ , u and v with $(\kappa^{<\kappa})^M = \kappa^{<\kappa}$. Let $\mathbb{P} = 2^{<\kappa}$ be the Cohen forcing. Suppose $M \preceq H(\kappa^+)$ is a model as above, i.e. transitive, $\kappa, u, v \in M$ and $(\kappa^{<\kappa})^M = \kappa^{<\kappa}$. Note that then $\mathbb{P} \cup \{\mathbb{P}\} \subset M$. Then, if G is \mathbb{P} -generic over M , then $\cup G \in D$ and there is ξ such that $\varphi(\cup G, \xi, u, v)$. By the definition of φ and u , an initial segment of ξ can be read from an initial segment of $\cup G$. That is why there is a nice \mathbb{P} -name τ for a function (see [19]) such that

$$\varphi(\cup G, \tau_G, u, v)$$

whenever G is \mathbb{P} -generic over M .

Now since the game EF_ω^κ is determined on all structures, (at least) one of the following holds:

- (1) there is p such that $p \Vdash \mathbf{II} \uparrow \text{EF}_\omega^\kappa(\mathcal{A}_\tau, \mathcal{A}_{r(\bar{0})})$,
- (2) there is p such that $p \Vdash \mathbf{I} \uparrow \text{EF}_\omega^\kappa(\mathcal{A}_\tau, \mathcal{A}_{r(\bar{0})})$

where $\bar{0}$ is the constant function with value 0. Let us show that both of them lead to a contradiction.

Assume (1). Fix a nice \mathbb{P} -name σ such that

$$p \Vdash \text{“}\sigma \text{ is a winning strategy of } \mathbf{II} \text{ in } \text{EF}_\omega^\kappa(\mathcal{A}_\tau, \mathcal{A}_{r(\bar{0})})\text{”}.$$

A strategy is a subset of $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa}$ (see Definition 5), and the forcing does not add elements to that set, so the nice name can be chosen such that all names in $\text{dom } \sigma$ are standard names for elements that are in $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa} \in H(\kappa^+)$.

Let M be an elementary submodel of $H(\kappa^+)$ of size κ such that

$$\{u, v, \sigma, r(\bar{0}), \tau, \mathbb{P}\} \cup (\kappa + 1) \cup M^{<\kappa} \subset M.$$

Listing all dense subsets of \mathbb{P} in M , it is easy to find a \mathbb{P} -generic G over M which contains p and such that $(\cup G)^{-1}\{1\}$ contains a cub. Now in V , $\cup G \Vdash \bar{0}$. Since $\varphi(\cup G, \tau_G, u, v)$ holds, we have by (∇) :

$$\mathcal{A}_{\tau_G} \not\cong \mathcal{A}_{r(\bar{0})}. \tag{i}$$

Let us show that σ_G is a winning strategy of player \mathbf{II} in $\text{EF}_\omega^\kappa(\mathcal{A}_{\tau_G}, \mathcal{A}_{r(\bar{0})})$ (in V) which by Theorem 77 above is a contradiction with (i).

Let μ be any strategy of player \mathbf{I} in $\text{EF}_\omega^\kappa(\mathcal{A}_{\tau_G}, \mathcal{A}_{r(\bar{0})})$ and let us show that σ_G beats it. Consider the play $\sigma_G * \mu$ and assume for a contradiction that it is a win for \mathbf{I} . This play is well defined, since the moves made by μ are in the domain of σ_G by the note after the definition of σ , and because $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa} \subset M$.

The play consists of ω moves and is a countable sequence in the set $([\kappa]^{<\kappa}) \times \kappa^{<\kappa}$ (see Definition of EF-games 5). Since \mathbb{P} is $< \kappa$ closed, there is $q_0 \in \mathbb{P}$ which decides $\sigma_G * \mu$ (i.e. $\sigma_{G_0} * \mu = \sigma_{G_1} * \mu$ whenever $q_0 \in G_0 \cap G_1$). Assume that G' is a \mathbb{P} -generic over V with $q_0 \in G'$. Then

$$(\sigma_{G'} * \mu)^{V[G']} = (\sigma_G * \mu)^{V[G']} = (\sigma_G * \mu)^V$$

(again, because \mathbb{P} does not add elements of $\kappa^{<\kappa}$) and so

$$(\sigma_{G'} * \mu \text{ is a win for } \mathbf{I})^{V[G']}.$$

But $q_0 \Vdash \text{“}\sigma * \mu \text{ is a win for } \mathbf{II}\text{”}$, because q_0 extends p and by the choice of σ .

The case (2) is similar, just instead of choosing $\cup G$ such that $(\cup G)^{-1}\{1\}$ contains a cub, choose G such that $(\cup G)^{-1}\{0\}$ contains a cub. Then we should have $\mathcal{A}_{\tau_G} \cong \mathcal{A}_{\tau(\bar{0})}$ which contradicts (2) by the same absoluteness argument as above. \square

6.2. Unstable and Superstable Theories. In this section we use Shelah’s ideas on how to prove non-structure theorems using Ehrenfeucht-Mostowski models, see [26]. We use the definition of Ehrenfeucht-Mostowski models from [14], Definition 4.2.

79. Definition. In the following discussion of linear orderings we use the following concepts.

- *Cointinality* or *reverse cofinality* of a linear order η , denoted $\text{cf}^*(\eta)$ is the smallest ordinal α such that there is a map $f: \alpha \rightarrow \eta$ which is strictly decreasing and $\text{ran } f$ has no (strict) lower bound in η .
- If $\eta = \langle \eta, < \rangle$ is a linear ordering, by η^* we denote its mirror image: $\eta^* = \langle \eta, <^* \rangle$ where $x <^* y \iff y < x$.
- Suppose λ is a cardinal. We say that an ordering η is λ -dense if for all subsets A and B of η with the properties $\forall a \in A \forall b \in B (a < b)$ and $|A| < \lambda$ and $|B| < \lambda$ there is $x \in \eta$ such that $a < x < b$ for all $a \in A, b \in B$. Dense means ω -dense.

80. Theorem. Suppose that $\kappa = \lambda^+ = 2^\lambda$ such that $\lambda^{<\lambda} = \lambda > \omega$. If T is unstable or superstable with OTOP or DOP, then $E_{S_\lambda^\kappa} \leq_c \cong_T$.

Proof. Note that from the cardinal assumptions it follows that $\lambda \geq 2^\omega$ and $(\lambda^+)^{\lambda^+} = \lambda^+$. We will carry out the proof for the case where T is unstable and shall make remarks on how certain steps of the proof should be modified

in order this to work for superstable theories with DOP or OTOP. First for each $S \subset S_\lambda^\kappa$, let us construct the linear orders $\Phi(S)$ which will serve a fundamental role in the construction. The following claim is Lemma 7.17 in [9]:

Claim 1. For each cardinal μ of uncountable cofinality there exists a linear ordering $\eta = \eta_\mu$ which satisfies:

- (1) $\eta \cong \eta + \eta$,
- (2) for all $\alpha \leq \mu$, $\eta \cong \eta \cdot \alpha + \eta$,
- (3) $\eta \cong \eta \cdot \mu + \eta \cdot \omega_1^*$,
- (4) η is dense,
- (5) $|\eta| = \mu$,
- (6) $\text{cf}^*(\eta) = \omega$.

Proof of Claim 1. Exactly as in [9].

□ Claim 1

For a set $S \subset S_\lambda^\kappa$, define the linear order $\Phi(S)$ as follows:

$$\Phi(S) = \sum_{i < \kappa} \tau(i, S),$$

where $\tau(i, S) = \eta_\lambda$ if $i \notin S$ and $\tau(i, S) = \eta_\lambda \cdot \omega_1^*$, if $i \in S$. Note that $\Phi(S)$ is dense and has size κ . For $\alpha < \beta < \kappa$ define

$$\Phi(S, \alpha, \beta) = \sum_{\alpha \leq i < \beta} \tau(i, S).$$

Then $\Phi(S, \alpha, \beta)$ has size $< \kappa$. (These definitions are also found in [9] although the idea dates back to J. Conway's Ph.D. thesis from the 1960's; they are first referred to in [23]). From now on denote $\eta = \eta_\lambda$.

Claim 2. If $\alpha \notin S$, then for all $\beta \geq \alpha$ we have $\Phi(S, \alpha, \beta + 1) \cong \eta$ and if $\alpha \in S$, then for all $\beta \geq \alpha$ we have $\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \omega_1^*$.

Proof of Claim 2. Let us begin by showing the first part, i.e. assume that $\alpha \notin S$. This is also like in [9]. We prove the statement by induction on $\text{OTP}(\beta \setminus \alpha)$. If $\beta = \alpha$, then $\Phi(S, \alpha, \alpha + 1) = \eta$ by the definition of Φ . If $\beta = \gamma + 1$ is a successor, then $\beta \notin S$, because S contains only limit ordinals, so $\tau(\beta, S) = \eta$ and

$$\Phi(S, \alpha, \beta + 1) = \Phi(S, \alpha, \gamma + 1 + 1) = \Phi(S, \alpha, \gamma + 1) + \eta$$

which by the induction hypothesis and by (1) is isomorphic to η . If $\beta \notin S$ is a limit ordinal, then choose a continuous cofinal sequence $s: \text{cf}(\beta) \rightarrow \beta$ such that

$s(\gamma) \notin S$ for all $\gamma < \text{cf}(\beta)$. This is possible since S contains only ordinals of cofinality λ . By the induction hypothesis $\Phi(S, \alpha, s(0) + 1) \cong \eta$,

$$\Phi(S, s(\gamma) + 1, s(\gamma + 1) + 1) \cong \eta$$

for all successor ordinals $\gamma < \text{cf}(\beta)$,

$$\Phi(S, s(\gamma), s(\gamma + 1) + 1) \cong \eta$$

for all limit ordinals $\gamma < \text{cf}(\beta)$ and so now

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \text{cf}(\beta) + \eta$$

which is isomorphic to η by (2). If $\beta \in S$, then $\text{cf}(\beta) = \lambda$ and we can again choose a cofinal sequence $s: \lambda \rightarrow \beta$ such that $s(\alpha)$ is not in S for all $\alpha < \lambda$. By the induction hypothesis. as above,

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \lambda + \tau(\beta, S)$$

and since $\beta \in S$ we have $\tau(\beta, S) = \eta \cdot \omega_1^*$, so we have

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \lambda + \eta \cdot \omega_1^*$$

which by (3) is isomorphic to η .

Suppose $\alpha \in S$. Then $\alpha + 1 \notin S$, so by the previous part we have

$$\Phi(S, \alpha, \beta + 1) \cong \tau(\alpha, S) + \Phi(S, \alpha + 1, \beta + 1) = \eta \cdot \omega_1^* + \eta = \eta \cdot \omega_1^*.$$

□ Claim 2

This gives us a way to show that the isomorphism type of $\Phi(S)$ depends only on the $E_{S_\lambda^\kappa}$ -equivalence class of S :

Claim 3. If $S, S' \subset S_\lambda^\kappa$ and $S \triangle S'$ is non-stationary, then $\Phi(S) \cong \Phi(S')$.

Proof of Claim 3. Let C be a cub set outside $S \triangle S'$. Enumerate it $C = \{\alpha_i \mid i < \kappa\}$ where $(\alpha_i)_{i < \kappa}$ is an increasing and continuous sequence. Now $\Phi(S) = \bigcup_{i < \kappa} \Phi(S, \alpha_i, \alpha_{i+1})$ and $\Phi(S') = \bigcup_{i < \kappa} \Phi(S', \alpha_i, \alpha_{i+1})$. Note that by the definitions these are disjoint unions, so it is enough to show that for all $i < \kappa$ the orders $\Phi(S, \alpha_i, \alpha_{i+1})$ and $\Phi(S', \alpha_i, \alpha_{i+1})$ are isomorphic. But for all $i < \kappa$ $\alpha_i \in S \iff \alpha_i \in S'$, so by Claim 2 either

$$\Phi(S, \alpha_i, \alpha_{i+1}) \cong \eta \cong \Phi(S', \alpha_i, \alpha_{i+1})$$

(if $\alpha_i \notin S$) or

$$\Phi(S, \alpha_i, \alpha_{i+1}) \cong \eta \cdot \omega_1^* \cong \Phi(S', \alpha_i, \alpha_{i+1})$$

(if $\alpha_i \in S$).

□ Claim 3

81. **Definition.** K_{tr}^λ is the set of L -models \mathcal{A} where $L = \{<, \leq, (P_\alpha)_{\alpha \leq \lambda}, h\}$, with the properties

- $\text{dom } \mathcal{A} \subset I^{\leq \lambda}$ for some linear order I .
- $\forall x, y \in A (x < y \iff x \subset y)$.
- $\forall x \in A (P_\alpha(x) \iff \text{length}(x) = \alpha)$.
- $\forall x, y \in A [x \leq y \iff \exists z \in A ((x, y \in \text{Succ}(z)) \wedge (I \models x < y))]$.
- $h(x, y)$ is the maximal common initial segment of x and y .

For each S , define the tree $T(S) \in K_{tr}^\lambda$ by

$$T(S) = \Phi(S)^{<\lambda} \cup \{\eta : \lambda \rightarrow \Phi(S) \mid \eta \text{ increasing and } \text{cf}^*(\Phi(S) \setminus \{x \mid (\exists y \in \text{ran } \eta)(x < y)\}) = \omega_1\}.$$

The relations $<$, \leq , P_n and h are interpreted in the natural way. From the cardinality assumptions it follows that $|T(S)| = \kappa$.

Clearly an isomorphism between $\Phi(S)$ and $\Phi(S')$ induces an isomorphism between $T(S)$ and $T(S')$, thus $T(S) \cong T(S')$ if $S \triangle S'$ is non-stationary.

Claim 4. Suppose T is unstable in the vocabulary v . Let T_1 be T with Skolem functions in the Skolemized vocabulary $v_1 \supset v$. Then there is a function $\mathcal{P}(S_\lambda^\kappa) \rightarrow \{\mathcal{A}^1 \mid \mathcal{A}^1 \models T_1, |\mathcal{A}^1| = \kappa\}$, $S \mapsto \mathcal{A}^1(S)$ which has following properties:

- (a) There is a mapping $T(S) \rightarrow (\text{dom } \mathcal{A}^1(S))^n$ for some $n < \omega$, $\eta \mapsto a_\eta$, such that $\mathcal{A}^1(S)$ is the Skolem hull of $\{a_\eta \mid \eta \in T(S)\}$, i.e. $\{a_\eta \mid \eta \in T(S)\}$ is the skeleton of $\mathcal{A}^1(S)$. Denote the skeleton of \mathcal{A} by $\text{Sk}(\mathcal{A})$.
- (b) $\mathcal{A}(S) = \mathcal{A}^1(S) \upharpoonright v$ is a model of T .
- (c) $\text{Sk}(\mathcal{A}^1(S))$ is indiscernible in $\mathcal{A}^1(S)$, i.e. if $\bar{\eta}, \bar{\xi} \in T(S)$ and $\text{tp}_{\text{q.f.}}(\bar{\eta}/\emptyset) = \text{tp}_{\text{q.f.}}(\bar{\xi}/\emptyset)$, then $\text{tp}(a_{\bar{\eta}}/\emptyset) = \text{tp}(a_{\bar{\xi}}/\emptyset)$ where $a_{\bar{\eta}} = (a_{\eta_1}, \dots, a_{\eta_{\text{length } \bar{\eta}}})$. This assignment of types in $\mathcal{A}^1(S)$ to q.f.-types in $T(S)$ is independent of S .
- (d) There is a formula $\varphi \in L_{\omega\omega}(v)$ such that for all $\eta, \nu \in T(S)$ and $\alpha < \lambda$, if $T(S) \models P_\lambda(\eta) \wedge P_\alpha(\nu)$, then $T(S) \models \eta > \nu$ if and only if $\mathcal{A}(S) \models \varphi(a_\eta, a_\nu)$.

Proof of Claim 4. The following is known:

- (F1) Suppose that T is a complete unstable theory. Then for each infinite linear order η , T has an Ehrenfeucht-Mostowski model \mathcal{A} , $|\mathcal{A}| = |\eta|$ of vocabulary v_1 , where $|v_1| = |T| + \omega$ and order is definable by a first-order formula, such that the template (assignment of types) is independent of η .²

²This is from [27]; there is a sketch of the proof also in [14], Theorem 4.7.

It is not hard to see that for every tree $t \in K_{tr}^\omega$ we can define a linear order $L(t)$ satisfying the following conditions:

- (1) $\text{dom}(L(t)) = (\text{dom } t \times \{0\}) \cup (\text{dom } t \times \{1\})$,
- (2) for all $a \in t$, $(a, 0) <_{L(t)} (a, 1)$,
- (3) if $a, b \in t$, then $a <_t b \iff [(a, 0) <_{L(t)} (b, 0)] \wedge [(b, 1) <_{L(t)} (a, 1)]$,
- (4) if $a, b \in t$, then

$$(a \not\leq b) \wedge (b \not\leq a) \iff [(b, 1) <_{L(t)} (a, 0)] \vee [(a, 1) <_{L(t)} (b, 0)].$$

Now for every $S \subset \kappa$, by (F1), there is an Ehrenfeucht-Mostowski model $\mathcal{A}^1(S)$ for the linear order $L(T(S))$ where order is definable by the formula ψ which is in $L_{\infty\omega}$. Suppose $\bar{\eta} = (\eta_0, \dots, \eta_n)$ and $\bar{\xi} = (\xi_0, \dots, \xi_n)$ are sequences in $T(S)$ that have the same quantifier free type. Then the sequences

$$\langle (\eta_0, 0), (\eta_0, 1), (\eta_1, 0), (\eta_1, 1), \dots, (\eta_n, 0), (\eta_n, 1) \rangle$$

and

$$\langle (\xi_0, 0), (\xi_0, 1), (\xi_1, 0), (\xi_1, 1), \dots, (\xi_n, 0), (\xi_n, 1) \rangle$$

have the same quantifier free type in $L(T(S))$ (refer to this property as $(\#)$). Now let the canonical skeleton of $\mathcal{A}^1(S)$ given by (F1) be $\{a_x \mid x \in L(T(S))\}$. Define the $T(S)$ -skeleton of $\mathcal{A}^1(S)$ to be the set

$$\{a_{(\eta,0)} \frown a_{(\eta,1)} \mid \eta \in T(S)\}.$$

Let us denote $b_\eta = a_{(\eta,0)} \frown a_{(\eta,1)}$. This guarantees that (a), (b) and (c) are satisfied.

For (d) suppose that the order $L(T(S))$ is definable in $\mathcal{A}(S)$ by the formula $\psi(\bar{u}, \bar{c})$, i.e. $\mathcal{A}(S) \models \psi(a_x, a_y) \iff x < y$ for $x, y \in L(T(S))$. Let $\varphi(x_0, x_1, y_0, y_1)$ be the formula

$$\psi(x_0, y_0) \wedge \psi(y_1, x_1).$$

Suppose $\eta, \nu \in T(S)$ are such that $T(S) \models P_\lambda(\eta) \wedge P_\alpha(\nu)$. Then

$$\varphi((a_\nu, 0), (a_\nu, 1), (a_\eta, 0), (a_\eta, 1))$$

holds in $\mathcal{A}(S)$ if and only if $\nu <_{T(S)} \eta$. □ Claim 4

Claim 5. Suppose $S \mapsto \mathcal{A}(S)$ is a function as described in Claim 4 with the identical notation. Suppose further that $S, S' \subset S_\lambda^\kappa$. Then $S \triangle S'$ is non-stationary if and only if $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Proof of Claim 5. Suppose $S \triangle S'$ is non-stationary. Then by Claim 3 $T(S) \cong$

$T(S')$ which implies $L(T(S)) \cong L(T(S'))$ (defined in the proof of Claim 4) which in turn implies $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Let us now show that if $S \triangle S'$ is stationary, then $\mathcal{A}(S) \not\cong \mathcal{A}(S')$. Let us make a counter assumption, namely that there is an isomorphism

$$f: \mathcal{A}(S) \cong \mathcal{A}(S')$$

and that $S \triangle S'$ is stationary, and let us deduce a contradiction. Without loss of generality we may assume that $S \setminus S'$ is stationary. Denote

$$X_0 = S \setminus S'.$$

For all $\alpha < \kappa$ define $T^\alpha(S)$ and $T^\alpha(S')$ by

$$T^\alpha(S) = \{\eta \in T(S) \mid \text{ran } \eta \subset \Phi(S, 0, \beta + 1) \text{ for some } \beta < \alpha\}$$

and

$$T^\alpha(S') = \{\eta \in T(S) \mid \text{ran } \eta \subset \Phi(S', 0, \beta + 1) \text{ for some } \beta < \alpha\}.$$

Then we have:

- (i) if $\alpha < \beta$, then $T^\alpha(S) \subset T^\beta(S)$,
- (ii) if γ is a limit ordinal, then $T^\gamma(S) = \bigcup_{\alpha < \gamma} T^\alpha(S)$.

The same of course holds for S' . Note that if $\alpha \in S \setminus S'$, then there is $\eta \in T^\alpha(S)$ cofinal in $\Phi(S, 0, \alpha)$ but there is no such $\eta \in T^\alpha(S')$ by definition of Φ : a cofinal function η is added only if $\text{cf}^*(\Phi(S', \alpha, \kappa)) = \omega_1$ which it is not if $\alpha \notin S'$. This is the key to achieving the contradiction.

But the clauses (i),(ii) are not sufficient to carry out the following argument, because we would like to have $|T^\alpha(S)| < \kappa$. That is why we want to define a different kind of filtration for $T(S)$, $T(S')$.

For all $\alpha \in X_0$ fix a function

$$\eta_\lambda^\alpha \in T(S) \tag{##}$$

such that $\text{dom } \eta_\lambda^\alpha = \lambda$, for all $\beta < \lambda$, $\eta_\lambda^\alpha \restriction \beta \in T^\alpha(S)$ and $\eta_\lambda^\alpha \notin T^\alpha(S)$.

For arbitrary $A \subset T(S) \cup T(S')$ let $\text{cl}_{\text{Sk}}(A)$ be the set $X \subset \mathcal{A}(S) \cup \mathcal{A}(S')$ such that $X \cap \mathcal{A}(S)$ is the Skolem closure of $\{a_\eta \mid \eta \in A \cap T(S)\}$ and $X \cap \mathcal{A}(S')$ the Skolem closure of $\{a_\eta \mid \eta \in A \cap T(S')\}$. The following is easily verified:

There exists a λ -cub set C and a set $K^\alpha \subset T^\alpha(S) \cup T^\alpha(S')$ for each $\alpha \in C$ such that

- (i') If $\alpha < \beta$, then $K^\alpha \subset K^\beta$.
- (ii') If γ is a limit ordinal in C , then $K^\gamma = \bigcup_{\alpha \in C \cap \gamma} K^\alpha$,

- (iii) for all $\beta < \alpha$, $\eta_\lambda^\beta \in K^\alpha$. (see $(\#\#)$ above).
- (iv) $|K^\alpha| = \lambda$.
- (v) $\text{cl}_{\text{Sk}}(K^\alpha)$ is closed under $f \cup f^{-1}$.
- (vi) $\{\eta \in T^\alpha(S) \cup T^\alpha(S') \mid \text{dom } \eta < \lambda\} \subset K^\alpha$.
- (vii) K^α is downward closed.

Denote $K^\kappa = \bigcup_{\alpha < \kappa} K^\alpha$. Clearly K^κ is closed under $f \cup f^{-1}$ and so f is an isomorphism between $\mathcal{A}(S) \cap \text{cl}_{\text{Sk}}(K^\kappa)$ and $\mathcal{A}(S') \cap \text{cl}_{\text{Sk}}(K^\kappa)$. We will derive a contradiction from this, i.e. we will actually show that $\mathcal{A}(S) \cap \text{cl}_{\text{Sk}}(K^\kappa)$ and $\mathcal{A}(S') \cap \text{cl}_{\text{Sk}}(K^\kappa)$ cannot be isomorphic by f . Clauses (iii), (v), (vi) and (vii) guarantee that all elements we are going to deal with will be in K^κ .

Let

$$X_1 = X_0 \cap C.$$

For $\alpha \in X_1$ let us use the following abbreviations:

- By $\mathcal{A}_\alpha(S)$ denote the Skolem closure of $\{a_\eta \mid \eta \in K^\alpha \cap T(S)\}$.
- By $\mathcal{A}_\alpha(S')$ denote the Skolem closure of $\{a_\eta \mid \eta \in K^\alpha \cap T(S')\}$.
- $K^\alpha(S) = K^\alpha \cap T(S)$.
- $K^\alpha(S') = K^\alpha \cap T(S')$.

In the following we will often deal with finite sequences. When defining such a sequence we will use a bar, but afterwards we will not use the bar in the notation (e.g. let $a = \bar{a}$ be a finite sequence...).

Suppose $\alpha \in X_1$. Choose

$$\xi_\lambda^\alpha = \bar{\xi}_\lambda^\alpha \in T(S') \tag{\#\#\#}$$

to be such that for some (finite sequence of) terms $\pi = \bar{\pi}$ we have

$$\begin{aligned} f(a_{\eta_\lambda^\alpha}) &= \pi(a_{\xi_\lambda^\alpha}) \\ &= \langle \pi_1(a_{\xi_\lambda^\alpha(1)}, \dots, a_{\xi_\lambda^\alpha(\text{length}(\bar{\xi}_\lambda^\alpha))}), \dots, \pi_{\text{length } \bar{\pi}}(a_{\xi_\lambda^\alpha(1)}, \dots, a_{\xi_\lambda^\alpha(\text{length}(\xi_\lambda^\alpha))}) \rangle. \end{aligned}$$

Note that ξ_λ^α is in K^κ by the definition of K^α 's.

$$\text{Let us denote by } \eta_\beta^\alpha, \text{ the element } \eta_\lambda^\alpha \upharpoonright \beta. \tag{\#\#\#\#}$$

Let

$$\xi_*^\alpha = \{\nu \in T(S') \mid \exists \xi \in \xi_\lambda^\alpha (\nu < \xi)\}.$$

Also note that $\xi_*^\alpha \subset K^\beta$ for some β .

Next define the function $g: X_1 \rightarrow \kappa$ as follows. Suppose $\alpha \in X_1$. Let $g(\alpha)$ be the smallest ordinal β such that $\xi_*^\alpha \cap K^\alpha(S') \subset K^\beta(S')$. We claim that

$g(\alpha) < \alpha$. Clearly $g(\alpha) \leq \alpha$, so suppose that $g(\alpha) = \alpha$. Since ξ_λ^α is finite, there must be a $\xi_\lambda^\alpha(i) \in \xi_\lambda^\alpha$ such that for all $\beta < \alpha$ there exists γ such that $\xi_\lambda^\alpha(i) \upharpoonright \gamma \in K^\alpha(S') \setminus K^\beta(S')$, i.e. $\xi_\lambda^\alpha(i)$ is cofinal in $\Phi(S', 0, \alpha)$ which it cannot be, because $\alpha \notin S'$.

Now by Fodor's lemma there exists a stationary set

$$X_2 \subset X_1$$

and γ_0 such that $g[X_2] = \{\gamma_0\}$.

Since there is only $< \kappa$ many finite sequences in $\mathcal{A}_{\gamma_0}(S')$, there is a stationary set

$$X_3 \subset X_2$$

and a finite sequence $\xi = \bar{\xi} \in K^{\gamma_0}(S')$ such that for all $\alpha \in X_3$ we have $\xi_\lambda^\alpha \cap K^{\gamma_0}(S') = \xi_*$ where ξ_* is the set

$$\xi_* = \{\nu \in T(S') \mid \nu \leq \zeta \text{ for some } \zeta \in \bar{\xi}\} \subset K^{\gamma_0}(S').$$

Let us fix a (finite sequence of) term(s) $\pi = \bar{\pi}$ such that the set

$$X_4 = \{\alpha \in X_3 \mid f(a_{\eta_\lambda^\alpha}) = \pi(a_{\xi_\lambda^\alpha})\}$$

is stationary (see ($\#\#$)). Here $f(\bar{a})$ means $\langle f(a_1), \dots, f(a_{\text{length } \bar{a}}) \rangle$ and $\bar{\pi}(\bar{b})$ means

$$\langle \pi_1(b_1, \dots, b_{\text{length } \bar{a}}), \dots, \pi_{\text{length } \pi}(b_1, \dots, b_{\text{length } \bar{a}}) \rangle.$$

We can find such π because there are only countably many such finite sequences of terms.

We claim that in $T(S')$ there are at most λ many quantifier free types over ξ_* . All types from now on are quantifier free. Let us show that there are at most λ many 1-types; the general case is left to the reader. To see this, note that a type p over ξ_* is described by the triple

$$(\nu_p, \beta_p, m_p) \tag{*}$$

defined as follows: if η satisfies p , then ν_p is the maximal element of ξ_* that is an initial segment of η , β_p is the level of η and m_p tells how many elements of $\xi_* \cap P_{\text{dom } \nu_p + 1}$ are there \leq -below $\eta(\text{dom } \nu_p)$ (recall the vocabulary from Definition 81).

Since $\nu_p \in \xi_*$ and ξ_* is of size λ , $\beta_p \in (\lambda + 1) \cup \{\infty\}$ and $m_p < \omega$, there can be at most λ such triples.

Recall the notations ($\#\#$), ($\#\#\#$) and ($\#\#\#\#$) above.

We can pick ordinals $\alpha < \alpha'$, $\alpha, \alpha' \in X_4$, a term τ and an ordinal $\beta < \lambda$ such that

$$\begin{aligned} \eta_\beta^{\alpha'} &\neq \eta_\beta^\alpha, \\ f(\eta_\beta^\alpha) &= \tau(a_{\xi_\beta^\alpha}) \text{ and } f(\eta_\beta^{\alpha'}) = \tau(a_{\xi_\beta^{\alpha'}}) \text{ for some } \xi_\beta^\alpha, \xi_\beta^{\alpha'}, \\ \text{tp}(\xi_\lambda^\alpha/\xi_*) &= \text{tp}(\xi_\lambda^{\alpha'}/\xi_*) \end{aligned}$$

and

$$\text{tp}(\xi_\beta^\alpha/\xi_*) = \text{tp}(\xi_\beta^{\alpha'}/\xi_*).$$

We claim that then in fact

$$\text{tp}(\xi_\beta^\alpha/(\xi_* \cup \{\xi_\beta^{\alpha'}\})) = \text{tp}(\xi_\beta^{\alpha'}/(\xi_* \cup \{\xi_\beta^\alpha\})).$$

Let us show this. Denote

$$p = \text{tp}(\xi_\beta^\alpha/(\xi_* \cup \{\xi_\lambda^{\alpha'}\}))$$

and

$$p' = \text{tp}(\xi_\beta^{\alpha'}/(\xi_* \cup \{\xi_\lambda^\alpha\})).$$

By the same reasoning as above at (\star) it is sufficient to show that these types p and p' have the same triple of the form (\star) . Since α and α' are in X_3 and X_2 , we have $\xi_*^{\alpha'} \cap K^{\alpha'}(S') = \xi_* \subset K^{\alpha'}(S')$. On the other hand $f \upharpoonright \mathcal{A}_{\alpha'}(S)$ is an isomorphism between $\mathcal{A}_{\alpha'}(S)$ and $\mathcal{A}_{\alpha'}(S')$, because α and α' are in X_1 , and so $\xi_\beta^{\alpha'} \in K^{\alpha'}(S')$. Thus $\nu_p = \nu_{p'} \in \xi_*$ and $m_p = m_{p'}$ follows in the same way. Clearly $\beta_p = \beta_{p'}$.

Now we have: ξ_λ^α and π are such that $f(\eta_\lambda^\alpha) = \pi(\xi_\lambda^\alpha)$ and ξ_β^α and τ are such that $f(\eta_\beta^\alpha) = \tau(\xi_\beta^\alpha)$. Similarly for α' . The formula φ is defined in Claim 4.

We know that

$$\mathcal{A}(S) \models \varphi(a_{\eta_\lambda^{\alpha'}}, a_{\eta_\beta^{\alpha'}})$$

and because f is isomorphism, this implies

$$\mathcal{A}(S') \models \varphi(f(a_{\eta_\lambda^{\alpha'}}), f(a_{\eta_\beta^{\alpha'}}))$$

which is equivalent to

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi_\lambda^{\alpha'}}), \tau(a_{\xi_\beta^{\alpha'}}))$$

(because α, α' are in X_4). Since $T(S')$ is indiscernible in $\mathcal{A}(S')$ and $\xi_\beta^{\alpha'}$ and ξ_β^α have the same type over $(\xi_* \cup \{\xi_\lambda^{\alpha'}\})$, we have

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi_\lambda^{\alpha'}}), \tau(a_{\xi_\beta^{\alpha'}})) \iff \varphi(\pi(a_{\xi_\lambda^{\alpha'}}), \tau(a_{\xi_\beta^\alpha})) \quad (*)$$

and so we get

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi_{\lambda}^{\alpha'}}), \tau(a_{\xi_{\beta}^{\alpha}}))$$

which is equivalent to

$$\mathcal{A}(S') \models \varphi(f(a_{\eta_{\lambda}^{\alpha'}}), f(a_{\eta_{\beta}^{\alpha}}))$$

and this in turn is equivalent to

$$\mathcal{A}(S) \models \varphi(a_{\eta_{\lambda}^{\alpha'}}, a_{\eta_{\beta}^{\alpha}}).$$

The latter cannot be true, because the definition of β, α and α' implies that $\eta_{\beta}^{\alpha'} \neq \eta_{\beta}^{\alpha}$. □ Claim 5

Thus, the above Claims 1 – 5 justify the embedding of $E_{S_{\lambda}^{\kappa}}$ into the isomorphism relation on the set of structures that are models for T for unstable T . This embedding combined with a suitable coding of models gives a continuous map.

DOP and OTOP cases. The above proof was based on the fact (F1) that for unstable theories there are Ehrenfeucht-Mostowski models for any linear order such that the order is definable by a first-order formula φ and is indiscernible relative to $L_{\omega\omega}$, (see (c) on page 79); it is used in (*) above. For the OTOP case, we use instead the fact (F2):

- (F2) Suppose that T is a theory with OTOP in a countable vocabulary v . Then for each dense linear order η we can find a model \mathcal{A} of a countable vocabulary $v_1 \supset v$ such that \mathcal{A} is an Ehrenfeucht-Mostowski model of T for η where order is definable by an $L_{\omega_1\omega}$ -formula.³

Since the order $\Phi(S)$ is dense, it is easy to argue that if $T(S)$ is indiscernible relative to $L_{\omega\omega}$, then it is indiscernible relative to $L_{\infty\omega}$ (define this as in (c) on page 79 changing tp to $\text{tp}_{L_{\infty\omega}}$). Other parts of the proof remain unchanged, because although the formula φ is not first-order anymore, it is still in $L_{\infty\omega}$.

In the DOP case we have the following fact:

- (F3) Let T be a countable superstable theory with DOP of vocabulary v . Then there exists a vocabulary $v_1 \supset v$, $|v_1| = \omega_1$, such that for every linear order η there exists a v_1 -model \mathcal{A} which is an Ehrenfeucht-Mostowski model of T for η where order is definable by an $L_{\omega_1\omega_1}$ -formula.⁴

³Contained in the proof of Theorem 2.5. of [24]; see also [14], Theorem 6.6.

⁴This is essentially from [28] Fact 2.5B; a proof can be found also in [14] Theorem 6.1.

Now the problem is that φ is in $L_{\infty\omega_1}$. By (c) of Claim 4, $T(S)$ is indiscernible in $\mathcal{A}(S)$ relative to $L_{\omega\omega}$ and by the above relative to $L_{\infty\omega}$. If we could require $\Phi(S)$ to be ω_1 -dense, we would similarly get indiscernible relative to $L_{\infty\omega_1}$. Let us show how to modify the proof in order to do that.

In Claim 1 (page 77), we have to replace clauses (3), (4) and (6) by (3'), (4') and (6'):

$$(3') \quad \eta \cong \eta \cdot \mu + \eta \cdot \omega^*,$$

$$(4') \quad \eta \text{ is } \omega_1\text{-dense},$$

$$(6') \quad \text{cf}^*(\eta) = \omega_1.$$

The proof that such an η exists is exactly as the proof of Lemma 7.17 [9] except that instead of putting $\mu = (\omega_1)^V$ put $\mu = \omega$, build θ -many functions with domains being countable initial segments of ω_1 instead of finite initial segments of ω and instead of \mathbb{Q} (the countable dense linear order) use an ω_1 -saturated dense linear order – this order has size 2^ω .

In the definition of $\Phi(S)$ (right after Claim 1), replace ω_1^* by ω^* and η by the new η satisfying (3'), (4') and (6') above. Note that $\Phi(S)$ becomes now ω_1 -dense. In Claim 2 one has to replace ω_1^* by ω^* . The proof remains similar. In the proof of Claim 3 (page 78) one has to adjust the use of Claim 2. Then, in the definition of $T(S)$ replace ω_1 by ω .

Claim 4 for superstable T with DOP now follows with (c) and (d) modified: instead of indiscernible relative to $L_{\omega\omega}$, demand $L_{\infty\omega_1}$ and instead of $\varphi \in L_{\omega\omega}$ we have now $\varphi \in L_{\infty\omega_1}$. The proof is unchanged except that the language is replaced by $L_{\infty\omega_1}$ everywhere and fact (F1) replaced by (F3) above.

Everything else in the proof, in particular the proof of Claim 5, remains unchanged modulo some obvious things that are evident from the above explanation. □ Theorem 80

6.3. Stable Unsuperstable Theories. In this section we provide a tree construction (Lemma 86) which is similar to Shelah's construction in [26] which he used to obtain (via Ehrenfeucht-Mostowski models) many pairwise non-isomorphic models. Then using a prime-model construction (proof of Theorem 87) we will obtain the needed result.

82. Definition. Let I be a tree of size κ . Suppose $(I_\alpha)_{\alpha < \kappa}$ is a collection of subsets of I such that

- For each $\alpha < \kappa$, I_α is a downward closed subset of I .
- $\bigcup_{\alpha < \kappa} I_\alpha = I$.

- If $\alpha < \beta < \kappa$, then $I_\alpha \subset I_\beta$.
- If γ is a limit ordinal, then $I_\gamma = \bigcup_{\alpha < \gamma} I_\alpha$.
- For each $\alpha < \kappa$ the cardinality of I_α is less than κ .

Such a sequence $(I_\alpha)_{\alpha < \kappa}$ is called κ -filtration or just filtration of I .

83. Definition. Recall K_{tr}^λ from Definition 81 on page 79. Let $K_{tr*}^\lambda = \{A \upharpoonright L^* \mid A \in K_{tr}^\lambda\}$, where L^* is the vocabulary $\{<\}$.

84. Definition. Suppose $t \in K_{tr*}^\omega$ is a tree of size κ (i.e. $t \subset \kappa^{<\omega}$) and let $\mathcal{I} = (I_\alpha)_{\alpha < \kappa}$ be a filtration of t . Define

$$S_{\mathcal{I}}(t) = \left\{ \alpha < \kappa \mid (\exists \eta \in t) [(\text{dom } \eta = \omega) \wedge \forall n < \omega (\eta \upharpoonright n \in I_\alpha) \wedge (\eta \notin I_\alpha)] \right\}.$$

By $S \sim_{\text{NS}} S'$ we mean that $S \triangle S'$ is not ω -stationary.

85. Lemma. Suppose trees t_0 and t_1 are isomorphic, and $\mathcal{I} = (I_\alpha)_{\alpha < \kappa}$ and $\mathcal{J} = (J_\alpha)_{\alpha < \kappa}$ are κ -filtrations of t_0 and t_1 respectively. Then $S_{\mathcal{I}}(t_0) \sim_{\text{NS}} S_{\mathcal{J}}(t_1)$.

Proof. Let $f: t_0 \rightarrow t_1$ be an isomorphism. Then $f\mathcal{I} = (f[I_\alpha])_{\alpha < \kappa}$ is a filtration of t_1 and

$$\alpha \in S_{\mathcal{I}}(t_0) \iff \alpha \in S_{f\mathcal{I}}(t_1). \quad (\star)$$

Define the set $C = \{\alpha \mid f[I_\alpha] = J_\alpha\}$. Let us show that it is cub. Let $\alpha \in \kappa$. Define $\alpha_0 = \alpha$ and by induction pick $(\alpha_n)_{n < \omega}$ such that $f[I_{\alpha_n}] \subset J_{\alpha_{n+1}}$ for odd n and $J_{\alpha_n} \subset f[I_{\alpha_{n+1}}]$ for even n . This is possible by the definition of a κ -filtration. Then $\alpha_\omega = \bigcup_{n < \omega} \alpha_n \in C$. Clearly C is closed and $C \subset \kappa \setminus S_{f\mathcal{I}}(t_1) \triangle S_{\mathcal{J}}(t_1)$, so now by (\star)

$$S_{\mathcal{I}}(t_0) = S_{f\mathcal{I}}(t_1) \sim_{\text{NS}} S_{\mathcal{J}}(t_1). \quad \square$$

86. Lemma. Suppose for $\lambda < \kappa$, $\lambda^\omega < \kappa$ and $\kappa^{<\kappa} = \kappa$. There exists a function $J: \mathcal{P}(\kappa) \rightarrow K_{tr*}^\omega$ such that

- $\forall S \subset \kappa (|J(S)| = \kappa)$.
- If $S \subset \kappa$ and \mathcal{I} is a κ filtration of $J(S)$, then $S_{\mathcal{I}}(J(S)) \sim_{\text{NS}} S$.
- If $S_0 \sim_{\text{NS}} S_1$, then $J(S_0) \cong J(S_1)$.

Proof. Let $S \subset S_\omega^\kappa$ and let us define a preliminary tree $I(S)$ as follows. For each $\alpha \in S$ let C_α be the set of all strictly increasing cofinal functions $\eta: \omega \rightarrow \alpha$. Let $I(S) = [\kappa]^{<\omega} \cup \bigcup_{\alpha \in S} C_\alpha$ where $[\kappa]^{<\omega}$ is the set of strictly increasing functions from finite ordinals to κ .

For ordinals $\alpha < \beta \leq \kappa$ and $i < \omega$ we adopt the notation:

- $[\alpha, \beta] = \{\gamma \mid \alpha \leq \gamma \leq \beta\}$
- $[\alpha, \beta) = \{\gamma \mid \alpha \leq \gamma < \beta\}$
- $\tilde{f}(\alpha, \beta, i) = \bigcup_{i \leq j \leq \omega} \{\eta \colon [i, j) \rightarrow [\alpha, \beta) \mid \eta \text{ strictly increasing}\}$

For each $\alpha, \beta < \kappa$ let us define the sets $P_\gamma^{\alpha, \beta}$, for $\gamma < \kappa$ as follows. If $\alpha = \beta = \gamma = 0$, then $P_0^{0,0} = I(S)$. Otherwise let $\{P_\gamma^{\alpha, \beta} \mid \gamma < \kappa\}$ enumerate all downward closed subsets of $\tilde{f}(\alpha, \beta, i)$ for all i , i.e.

$$\{P_\gamma^{\alpha, \beta} \mid \gamma < \kappa\} = \bigcup_{i < \omega} \mathcal{P}(\tilde{f}(\alpha, \beta, i)) \cap \{A \mid A \text{ is closed under initial segments}\}.$$

Define

$$\tilde{n}(P_\gamma^{\alpha, \beta})$$

to be the natural number i such that $P_\gamma^{\alpha, \beta} \subset \tilde{f}(\alpha, \beta, i)$. The enumeration is possible, because by our assumption $\kappa^{<\kappa} = \kappa$ we have

$$\begin{aligned} \left| \bigcup_{i < \omega} \mathcal{P}(\tilde{f}(\alpha, \beta, i)) \right| &\leq \omega \times |\mathcal{P}(\tilde{f}(0, \beta, 0))| \\ &\leq \omega \times |\mathcal{P}(\beta^\omega)| \\ &= \omega \times 2^{\beta^\omega} \\ &\leq \omega \times \kappa \\ &= \kappa \end{aligned}$$

Let $S \subset \kappa$ be a set and define $J(S)$ to be the set of all $\eta \colon s \rightarrow \omega \times \kappa^4$ such that $s \leq \omega$ and the following conditions are met for all $i, j < s$:

- (1) η is strictly increasing with respect to the lexicographical order on $\omega \times \kappa^4$.
- (2) $\eta_1(i) \leq \eta_1(i+1) \leq \eta_1(i) + 1$
- (3) $\eta_1(i) = 0 \rightarrow \eta_2(i) = \eta_3(i) = \eta_4(i) = 0$.
- (4) $\eta_1(i) < \eta_1(i+1) \rightarrow \eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$.
- (5) $\eta_1(i) = \eta_1(i+1) \rightarrow (\forall k \in \{2, 3, 4\})(\eta_k(i) = \eta_k(i+1))$.
- (6) if for some $k < \omega$, $[i, j) = \eta_1^{-1}\{k\}$, then $\eta_5 \upharpoonright [i, j) \in P_{\eta_4(i)}^{\eta_2(i), \eta_3(i)}$.
- (7) if $s = \omega$, then either $(\exists m < \omega)(\forall k < \omega)(k > m \rightarrow \eta_1(k) = \eta_1(k+1))$ or $\sup \text{ran } \eta_5 \in S$.
- (8) Order $J(S)$ by inclusion.

Note that it follows from the definition of $P_\gamma^{\alpha,\beta}$ and the conditions (6) and (4) that for all $i < j < \text{dom } \eta$, $\eta \in J(S)$:

$$(9) \quad i < j \rightarrow \eta_5(i) < \eta_5(j).$$

For each $\alpha < \kappa$ let

$$J^\alpha(S) = \{\eta \in J(S) \mid \text{ran } \eta \subset \omega \times (\beta + 1)^4 \text{ for some } \beta < \alpha\}.$$

Then $(J^\alpha(S))_{\alpha < \kappa}$ is a κ -filtration of $J(S)$ (see Claim 2 below). For the first item of the lemma, clearly $|J(S)| = \kappa$.

Let us observe that if $\eta \in J(S)$ and $\text{ran } \eta_1 = \omega$, then

$$\sup \text{ran } \eta_4 \leq \sup \text{ran } \eta_2 = \sup \text{ran } \eta_3 = \sup \text{ran } \eta_5 \quad (\#)$$

and if in addition to that, $\eta \upharpoonright k \in J^\alpha(S)$ for all k and $\eta \notin J^\alpha(S)$ or if $\text{ran } \eta_1 = \{0\}$, then

$$\sup \text{ran } \eta_5 = \alpha. \quad (\circledast)$$

To see $(\#)$ suppose $\text{ran } \eta_1 = \omega$. By (9), $(\eta_5(i))_{i < \omega}$ is an increasing sequence. By (6) $\sup \text{ran } \eta_3 \geq \sup \text{ran } \eta_5 \geq \sup \text{ran } \eta_2$. By (4), $\sup \text{ran } \eta_2 \geq \sup \text{ran } \eta_3$ and again by (4) $\sup \text{ran } \eta_2 \geq \sup \text{ran } \eta_4$. Inequality $\sup \text{ran } \eta_5 \leq \alpha$ is an immediate consequence of the definition of $J^\alpha(S)$, so (\circledast) follows now from the assumption that $\eta \notin J^\alpha(S)$.

Claim 1. Suppose $\xi \in J^\alpha(S)$ and $\eta \in J(S)$. Then if $\text{dom } \xi < \omega$, $\xi \subsetneq \eta$ and $(\forall k \in \text{dom } \eta \setminus \text{dom } \xi)(\eta_1(k) = \xi_1(\max \text{dom } \xi) \wedge \eta_1(k) > 0)$, then $\eta \in J^\alpha(S)$.

Proof of Claim 1. Suppose $\xi, \eta \in J^\alpha(S)$ are as in the assumption. Let us define $\beta_2 = \xi_2(\max \text{dom } \xi)$, $\beta_3 = \xi_3(\max \text{dom } \xi)$, and $\beta_4 = \xi_4(\max \text{dom } \xi)$. Because $\xi \in J^\alpha(S)$, there is β such that $\beta_2, \beta_3, \beta_4 < \beta + 1$ and $\beta < \alpha$. Now by (5) $\eta_2(k) = \beta_2$, $\eta_3(k) = \beta_3$ and $\eta_4(k) = \beta_4$, for all $k \in \text{dom } \eta \setminus \text{dom } \xi$. Then by (6) for all $k \in \text{dom } \eta \setminus \text{dom } \xi$ we have that $\beta_2 < \eta_5(k) < \beta_3 < \beta + 1$. Since $\xi \in J^\alpha(S)$, also $\beta_4 < \beta + 1$, so $\eta \in J^\alpha(S)$. \square Claim 1

Claim 2. $|J(S)| = \kappa$, $(J^\alpha(S))_{\alpha < \kappa}$ is a κ -filtration of $J(S)$ and if $S \subset \kappa$ and \mathcal{I} is a κ -filtration of $J(S)$, then $S_{\mathcal{I}}(J(S)) \sim_{\text{NS}} S$.

Proof of Claim 2. For all $\alpha \leq \kappa$, $J^\alpha(S) \subset (\omega \times \alpha^4)^{\leq \omega}$, so by the cardinality assumption of the lemma, the cardinality of $J^\alpha(S)$ is $< \kappa$ if $\alpha < \kappa$ ($J^\kappa(S) = J(S)$).

Clearly $\alpha < \beta$ implies $J^\alpha(S) \subset J^\beta(S)$. Continuity is verified by

$$\begin{aligned} \bigcup_{\alpha < \gamma} J^\alpha(S) &= \{\eta \in J(S) \mid \exists \alpha < \gamma, \exists \beta < \alpha (\text{ran } \eta \subset \omega \times (\beta + 1)^4)\} \\ &= \{\eta \in J(S) \mid \exists \beta < \cup \gamma (\text{ran } \eta \subset \omega \times (\beta + 1)^4)\} \end{aligned}$$

which equals $J^\gamma(S)$ if γ is a limit ordinal. By Lemma 85 it is enough to show $S_{\mathcal{I}}(J(S)) \sim_{\text{NS}} S$ for $\mathcal{I} = (J^\alpha(S))_{\alpha < \kappa}$, and we will show that if $\mathcal{I} = (J^\alpha(S))_{\alpha < \kappa}$, then in fact $S_{\mathcal{I}}(J(S)) = S$.

Suppose $\alpha \in S_{\mathcal{I}}(J(S))$. Then there is $\eta \in J(S)$, $\text{dom } \eta = \omega$, such that $\eta \restriction k \in J^\alpha(S)$ for all $k < \omega$ but $\eta \notin J^\alpha(S)$. Thus there is no $\beta < \alpha$ such that $\text{ran } \eta \subset \omega \times (\beta + 1)^4$ but on the other hand for all $k < \omega$ there is β such that $\text{ran } \eta \restriction k \subset \omega \times (\beta + 1)^4$. By (5) and (6) this implies that either $\text{ran } \eta_1 = \omega$ or $\text{ran } \eta_1 = \{0\}$. By $(*)$ on page 89 it now follows that $\sup \text{ran } \eta_5 = \alpha$ and by (7), $\alpha \in S$.

Suppose then that $\alpha \in S$. Let us show that $\alpha \in S_{\mathcal{I}}(J(S))$. Fix a function $\eta_\alpha: \omega \rightarrow \kappa$ with $\sup \text{ran } \eta_\alpha = \alpha$. Then $\eta_\alpha \in I(S)$ and the function η such that $\eta(n) = (0, 0, 0, 0, \eta_\alpha(n))$ is as required. (Recall that $P_0^{0,0} = I(S)$ in the definition of $J(S)$). \square Claim 2

Claim 3. Suppose $S \sim_{\text{NS}} S'$. Then $J(S) \cong J(S')$.

Proof of Claim 3. Let $C \subset \kappa \setminus (S \triangle S')$ be the cub set which exists by the assumption. By induction on $i < \kappa$ we will define α_i and F_{α_i} such that

- (a) If $i < j < \kappa$, then $\alpha_i < \alpha_j$ and $F_{\alpha_i} \subset F_{\alpha_j}$.
- (b) If i is a successor, then α_i is a successor and if i is limit, then $\alpha_i \in C$.
- (c) If γ is a limit ordinal, then $\alpha_\gamma = \sup_{i < \gamma} \alpha_i$,
- (d) F_{α_i} is a partial isomorphism $J(S) \rightarrow J(S')$
- (e) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0 and $n < \omega$ is even. Then $\text{dom } F_{\alpha_i} = J^{\alpha_i}(S)$ (e1). If also $n > 0$ and $(\eta_k)_{k < \omega}$ is an increasing sequence in $J^{\alpha_i}(S)$ such that $\eta = \bigcup_{k < \omega} \eta_k \notin J(S)$, then $\bigcup_{k < \omega} F_{\alpha_i}(\eta_k) \notin J(S')$ (e2).
- (f) If $i = \gamma + n$, where γ is a limit ordinal or 0 and $n < \omega$ is odd, then $\text{ran } F_{\alpha_i} = J^{\alpha_i}(S')$ (f1). Further, if $(\eta_k)_{k < \omega}$ is an increasing sequence in $J^{\alpha_i}(S')$ such that $\eta = \bigcup_{k < \omega} \eta_k \notin J(S')$, then $\bigcup_{k < \omega} F_{\alpha_i}^{-1}(\eta_k) \notin J(S)$ (f3).
- (g) If $\text{dom } \xi < \omega$, $\xi \in \text{dom } F_{\alpha_i}$, $\eta \restriction \text{dom } \xi = \xi$ and $(\forall k \geq \text{dom } \xi)(\eta_1(k) = \xi_1(\max \text{dom } \xi) \wedge \eta_1(k) > 0)$, then $\eta \in \text{dom } F_{\alpha_i}$. Similarly for $\text{ran } F_{\alpha_i}$.
- (h) If $\xi \in \text{dom } F_{\alpha_i}$ and $k < \text{dom } \xi$, then $\xi \restriction k \in \text{dom } F_{\alpha_i}$.
- (i) For all $\eta \in \text{dom } F_{\alpha_i}$, $\text{dom } \eta = \text{dom}(F_{\alpha_i}(\eta))$.

The first step. The first step and the successor steps are similar, but the first step is easier. Thus we give it separately in order to simplify the readability. Let us start with $i = 0$. Let $\alpha_0 = \beta + 1$, for arbitrary $\beta \in C$. Let us denote by

$$\tilde{o}(\alpha)$$

the ordinal that is order isomorphic to $(\omega \times \alpha^4, <_{\text{lex}})$. Let γ be such that there is an isomorphism $h: P_\gamma^{0, \tilde{o}(\alpha_0)} \cong J^{\alpha_0}(S)$ and such that $\tilde{n}(P_\gamma^{0, \alpha_0}) = 0$. Such exists by (1). Suppose that $\eta \in J^{\alpha_0}(S)$. Note that because P_γ^{0, α_0} and $J^{\alpha_0}(S)$ are closed under initial segments and by the definitions of \tilde{n} and $P_\gamma^{\alpha, \beta}$, we have $\text{dom } h^{-1}(\eta) = \text{dom } \eta$. Define $\xi = F_{\alpha_0}(\eta)$ such that $\text{dom } \xi = \text{dom } \eta$ and for all $k < \text{dom } \xi$

- $\xi_1(k) = 1$
- $\xi_2(k) = 0$
- $\xi_3(k) = \tilde{o}(\alpha_0)$
- $\xi_4(k) = \gamma$
- $\xi_5(k) = h^{-1}(\eta)(k)$.

Let us check that $\xi \in J(S')$. Conditions (1)-(5) and (7) are satisfied because ξ_k is constant for all $k \in \{1, 2, 3, 4\}$, $\xi_1(i) \neq 0$ for all i and ξ_5 is increasing. For (6), if $\xi_1^{-1}\{k\}$ is empty, the condition is verified since each $P_\gamma^{\alpha, \beta}$ is closed under initial segments and contains the empty function. If it is non-empty, then $k = 1$ and in that case $\xi_1^{-1}\{k\} = [0, \omega)$ and by the argument above ($\text{dom } h^{-1}(\eta) = \text{dom } \eta = \text{dom } \xi$) we have $\xi_5 = h^{-1}(\eta) \in P_\gamma^{0, \tilde{o}(\alpha_0)} = P_{\xi_4(0)}^{\xi_2(0), \xi_3(0)}$, so the condition is satisfied.

Let us check whether all the conditions (a)-(i) are met. In (a), (b), (c), (e2) and (f) there is nothing to check. (d) holds, because h is an isomorphism. (e1) and (i) are immediate from the definition. Both $J^{\alpha_0}(S)$ and $P_\gamma^{0, \tilde{o}(\alpha_0)}$ are closed under initial segments, so (h) follows, because $\text{dom } F_{\alpha_0} = J^{\alpha_0}(S)$ and $\text{ran } F_{\alpha_0} = \{1\} \times \{0\} \times \{\tilde{o}(\alpha_0)\} \times \{\gamma\} \times P_\gamma^{0, \alpha_0}$. Claim 1 implies (g) for $\text{dom } F_{\alpha_0}$. Suppose $\xi \in \text{ran } F_{\alpha_0}$ and $\eta \in J(S')$ are as in the assumption of (g). Then $\eta_1(i) = \xi_1(i) = 1$ for all $i < \text{dom } \eta$. By (5) it follows that $\eta_2(i) = \xi_2(i) = 0$, $\eta_3(i) = \xi_3(i) = \tilde{o}(\alpha_0)$ and $\eta_4(i) = \xi_4(i) = \gamma$ for all $i < \text{dom } \eta$, so by (6) $\eta_5 \in P_\gamma^{0, \tilde{o}(\alpha_0)}$ and since h is an isomorphism, $\eta \in \text{ran } F_{\alpha_0}$.

Odd successor step. We want to handle odd case but not the even case first, because the most important case is the successor of a limit ordinal, see ($\mu\mu$) below. Except that, the even case is similar to the odd case.

Suppose that $j < \kappa$ is a successor ordinal. Then there exist β_j and n_j such that $j = \beta_j + n_j$ and β_j is a limit ordinal or 0. Suppose that n_j is odd and that

α_l and F_{α_l} are defined for all $l < j$ such that the conditions (a)–(i) and (1)–(9) hold for $l < j$.

Let $\alpha_j = \beta + 1$ where β is such that $\beta \in C$, $\text{ran } F_{\alpha_{j-1}} \subset J^\beta(S')$, $\beta > \alpha_{j-1}$. For convenience define $\xi(-1) = (0, 0, 0, 0, 0)$ for all $\xi \in J(S) \cup J(S')$. Suppose $\eta \in \text{ran } F_{\alpha_{j-1}}$ has finite domain $\text{dom } \eta = m < \omega$ and denote $\xi = F_{\alpha_{j-1}}^{-1}(\eta)$. Fix γ_η to be such that $\tilde{n}(P_{\gamma_\eta}^{\alpha, \beta}) = m$ and such that there is an isomorphism $h_\eta: P_{\gamma_\eta}^{\alpha, \beta} \rightarrow W$, where

$$W = \{\zeta \mid \text{dom } \zeta = [m, s), m < s \leq \omega, \eta \frown \langle m, \zeta(m) \rangle \notin \text{ran } F_{\alpha_{j-1}}, \eta \frown \zeta \in J^{\alpha_j}(S')\},$$

$\alpha = \xi_3(m-1) + \xi_4(m-1)$ and $\beta = \alpha + \tilde{o}(\alpha_j)$ (defined in the beginning of the First step).

We will define F_{α_j} so that its range is $J^{\alpha_j}(S')$ and instead of F_{α_j} we will define its inverse. So let $\eta \in J^{\alpha_j}(S')$. We have three cases:

- (ι) $\eta \in \text{ran } F_{\alpha_{j-1}}$,
- ($\iota\iota$) $\exists m < \text{dom } \eta (\eta \upharpoonright m \in \text{ran } F_{\alpha_{j-1}} \wedge \eta \upharpoonright (m+1) \notin F_{\alpha_{j-1}})$,
- ($\iota\iota\iota$) $\forall m < \text{dom } \eta (\eta \upharpoonright (m+1) \in \text{ran } F_{\alpha_{j-1}} \wedge \eta \notin \text{ran } F_{\alpha_{j-1}})$.

Let us define $\xi = F_{\alpha_j}^{-1}(\eta)$ such that $\text{dom } \xi = \text{dom } \eta$. If (ι) holds, define $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta)(n)$ for all $n < \text{dom } \eta$. Clearly $\xi \in J(S)$ by the induction hypothesis. Suppose that ($\iota\iota$) holds and let m witness this. For all $n < \text{dom } \xi$ let

- If $n < m$, then $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta \upharpoonright m)(n)$.
- Suppose $n \geq m$. Let
 - $\xi_1(n) = \xi_1(m-1) + 1$
 - $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$
 - $\xi_3(n) = \xi_2(m) + \tilde{o}(\alpha_j)$
 - $\xi_4(n) = \gamma_{\eta \upharpoonright m}$
 - $\xi_5(n) = h_{\eta \upharpoonright m}^{-1}(\eta)(n)$.

Next we should check that $\xi \in J(S)$; let us check items (1) and (6), the rest are left to the reader.

- (1) By the induction hypothesis $\xi \upharpoonright m$ is increasing. Next, $\xi_1(m) = \xi_1(m-1) + 1$, so $\xi(m-1) <_{\text{lex}} \xi(m)$. If $m \leq n_1 < n_2$, then $\xi_k(n_1) = \xi_k(n_2)$ for all $k \in \{1, 2, 3, 4\}$ and ξ_5 is increasing.
- (6) Suppose that $[i, j) = \xi_1^{-1}\{k\}$. Since $\xi_1 \upharpoonright [m, \omega)$ is constant, either $j < m$, when we are done by the induction hypothesis, or $i = m$ and $j = \omega$. In that case one verifies that $\eta \upharpoonright [m, \omega) \in W = \text{ran } h_{\eta \upharpoonright m}$ and then, imitating the corresponding

argument in the first step, that

$$\xi_5 \upharpoonright [m, \omega) = h_{\eta \upharpoonright m}^{-1}(\eta \upharpoonright [m, \omega))$$

and hence in $\text{dom } h_{\eta \upharpoonright m} = P_{\xi_4(m)}^{\xi_2(m), \xi_3(m)}$.

Suppose finally that $(\iota\iota)$ holds. Then $\text{dom } \eta$ must be ω since otherwise the condition $(\iota\iota)$ is simply contradictory (because $\eta \upharpoonright (\text{dom } \eta - 1 + 1) = \eta$ (except for the case $\text{dom } \eta = 0$, but then condition (ι) holds and we are done)). By (g), we have $\text{ran } \eta_1 = \omega$, because otherwise we had $\eta \in \text{ran } F_{\alpha_{j-1}}$. Let $F_{\alpha_j}^{-1}(\eta) = \xi = \bigcup_{n < \omega} F_{\alpha_{j-1}}^{-1}(\eta \upharpoonright n)$.

Let us check that it is in $J(S)$. Conditions (1)–(6) are satisfied by ξ , because they are satisfied by all its initial segments. Let us check (7).

First of all ξ cannot be in $J^{\alpha_{j-1}}(S)$, since otherwise, by (d) and (i),

$$F_{\alpha_{j-1}}(\xi) = \bigcup_{n < \omega} F_{\alpha_{j-1}}(\xi \upharpoonright n) = \bigcup_{n < \omega} \eta \upharpoonright n = \eta$$

were again in $\text{ran } F_{\alpha_{j-1}}$. If $j - 1$ is a successor ordinal, then we are done: by (b) α_{j-1} is a successor and we assumed $\eta \in J(S')$, so by (e2) we have $\xi \in J(S)$. Thus we can assume that $j - 1$ is a limit ordinal. Then by (b), α_{j-1} is a limit ordinal in C and by (a), (e) and (f), $\text{ran } F_{\alpha_{j-1}} = J^{\alpha_{j-1}}(S')$ and $\text{dom } F_{\alpha_{j-1}} = J^{\alpha_{j-1}}(S)$. This implies that $\text{ran } \eta \not\subseteq \omega \times \beta^4$ for any $\beta < \alpha_{j-1}$ and by (\otimes) on page 89 we must have $\sup \text{ran } \eta_5 = \alpha_{j-1}$ which gives $\alpha_{j-1} \in S'$ by (7). Since $\alpha_{j-1} \in C \subset \kappa \setminus S \triangle S'$, we have $\alpha_{j-1} \in S$. Again by (\otimes) and that $\text{dom } F_{\alpha_{j-1}} = J^{\alpha_{j-1}}(S)$ by (e1), we have $\sup \text{ran } \xi_5 = \alpha_{j-1}$, thus ξ satisfies the condition (7).

Let us check whether all the conditions (a)–(i) are met. (a), (b), (c) are common to the cases (ι) , $(\iota\iota)$ and $(\iota\iota\iota)$ in the definition of $F_{\alpha_j}^{-1}$ and are easy to verify. Let us sketch a proof for (d); the rest is left to the reader.

(d) Let $\eta_1, \eta_2 \in \text{ran } F_{\alpha_j}$ and let us show that

$$\eta_1 \subsetneq \eta_2 \iff F_{\alpha_j}^{-1}(\eta_1) \subsetneq F_{\alpha_j}^{-1}(\eta_2).$$

The case where both η_1 and η_2 satisfy $(\iota\iota)$ is the interesting one (implies all the others).

So suppose $\eta_1, \eta_2 \in (\iota\iota)$. Then there exist m_1 and m_2 as described in the statement of $(\iota\iota)$. Let us show that $m_1 = m_2$. We have $\eta_1 \upharpoonright (m_1 + 1) = \eta_2 \upharpoonright (m_1 + 1)$ and $\eta_1 \upharpoonright (m_1 + 1) \notin \text{ran } F_{\alpha_{j-1}}$, so $m_2 \leq m_1$. If $m_2 \leq m_1$, then $m_2 < \text{dom } \eta_1$, since $m_1 < \text{dom } \eta_1$. Thus if $m_2 \leq m_1$, then $\eta_1 \upharpoonright (m_2 + 1) = \eta_2 \upharpoonright (m_2 + 1) \notin \text{ran } F_{\alpha_{j-1}}$, which implies $m_2 = m_1$. According to the definition

of $F_{\alpha_j}^{-1}(\eta_i)(k)$ for $k < \text{dom } \eta_1$, $F_{\alpha_j}^{-1}(\eta_i)(k)$ depends only on m_i and $\eta \restriction m_i$ for $i \in \{1, 2\}$. Since $m_1 = m_2$ and $\eta_1 \restriction m_1 = \eta_2 \restriction m_2$, we have $F_{\alpha_j}^{-1}(\eta_1)(k) = F_{\alpha_j}^{-1}(\eta_2)(k)$ for all $k < \text{dom } \eta_1$.

Let us now assume that $\eta_1 \not\subseteq \eta_2$. Then take the smallest $n \in \text{dom } \eta_1 \cap \text{dom } \eta_2$ such that $\eta_1(n) \neq \eta_2(n)$. It is now easy to show that $F_{\alpha_j}^{-1}(\eta_1)(n) \neq F_{\alpha_j}^{-1}(\eta_2)(n)$ by the construction.

Even successor step. Namely the one where $j = \beta + n$ and n is even. But this case goes exactly as the above completed step, except that we start with $\text{dom } F_{\alpha_j} = J^{\alpha_j}(S)$ where α_j is big enough successor of an element of C such that $J^{\alpha_j}(S)$ contains $\text{ran } F_{\alpha_{j-1}}$ and define $\xi = F_{\alpha_j}(\eta)$. Instead of (e) we use (f) as the induction hypothesis. This step is easier since one does not need to care about the successors of limit ordinals.

Limit step. Assume that j is a limit ordinal. Then let $\alpha_j = \bigcup_{i < j} \alpha_i$ and $F_{\alpha_j} = \bigcup_{i < j} F_{\alpha_i}$. Since α_i are successors of ordinals in C , $\alpha_j \in C$, so (b) is satisfied. Since each F_{α_i} is an isomorphism, also their union is, so (d) is satisfied. Because conditions (e), (f) and (i) hold for $i < j$, the conditions (e) and (i) hold for j . (f) is satisfied because the premise is not true. (a) and (c) are clearly satisfied. Also (g) and (h) are satisfied by Claim 1 since now $\text{dom } F_{\alpha_j} = J^{\alpha_j}(S)$ and $\text{ran } F_{\alpha_j} = J^{\alpha_j}(S')$ (this is because (a), (e) and (f) hold for $i < j$).

Finally $F = \bigcup_{i < \kappa} F_{\alpha_i}$ is an isomorphism between $J(S)$ and $J(S')$. □ Claim 3

□ Lemma 86

87. Theorem. Suppose κ is such that $\kappa^{<\kappa} = \kappa$ and for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and that T is a stable unsuperstable theory. Then $E_{S_\kappa^\omega} \leq_c \cong_T$.

Proof. For $\eta \in 2^\kappa$ let $J_\eta = J(\eta^{-1}\{1\})$ where the function J is as in Lemma 86 above. For notational convenience, we assume that J_η is a downward closed subtree of $\kappa^{\leq \omega}$. Since T is stable unsuperstable, for all η and $t \in J_\eta$, there are finite sequences $a_t = a_t^\eta$ in the monster model such that

(1) If $\text{dom}(t) = \omega$ and $n < \omega$ then

$$\begin{array}{ccc} a_t & \searrow & a_{t \restriction n} \\ & \bigcup_{m < n} & \\ & a_t \restriction m & \end{array}$$

(2) For all downward closed subtrees $X, Y \subset J_\eta$,

$$\bigcup_{t \in X} a_t \quad \downarrow \quad \bigcup_{t \in Y} a_t.$$

$\bigcup_{t \in X \cap Y} a_t$

- (3) For all downward closed subtrees $X \subset J_\eta$ and $Y \subset J_{\eta'}$ the following holds:
 If $f: X \rightarrow Y$ is an isomorphism, then there is an automorphism F of the monster model such that for all $t \in X$, $F(a_t^\eta) = a_{f(t)}^{\eta'}$.

Then we can find an F_ω^f -construction

$$(\bigcup_{t \in J_\eta} a_t, (b_i, B_i)_{i < \kappa})$$

(here $(t(b/C), D) \in F_\omega^f$ if $D \subset C$ is finite and $b \downarrow_D C$, see [25]) such that

- (\star) for all $\alpha < \kappa$, c and finite $B \subset \bigcup_{t \in J_\eta} a_t \cup \bigcup_{i < \alpha} b_i$ there is $\alpha < \beta < \kappa$ such that $B_\beta = B$ and

$$\text{stp}(b_\beta/B) = \text{stp}(c/B).$$

Then

$$M_\eta = \bigcup_{t \in J_\eta} a_t \cup \bigcup_{i < \kappa} b_i \models T.$$

Without loss of generality we may assume that the trees J_η and the F_ω^f -constructions for M_η are chosen coherently enough such that one can find a code ξ_η for (the isomorphism type of) M_η so that $\eta \mapsto \xi_\eta$ is continuous. Thus we are left to show that $\eta E_{S_\omega^\kappa} \eta' \iff M_\eta \cong M_{\eta'}$

“ \implies ” Assume $J_\eta \cong J_{\eta'}$. By (3) it is enough to show that F_ω^f -construction of length κ satisfying (\star) are unique up to isomorphism over $\bigcup_{t \in J_\eta} a_t$. But (\star) guarantees that the proof of the uniqueness of F -primary models from [25] works here.

“ \impliedby ” Suppose $F: M_\eta \rightarrow M_{\eta'}$ is an isomorphism and for a contradiction suppose $(\eta, \eta') \notin E_{S_\omega^\kappa}$. Let $(J_\eta^\alpha)_{\alpha < \kappa}$ be a filtration of J_η and $(J_{\eta'}^\alpha)_{\alpha < \kappa}$ be a filtration of $J_{\eta'}$ (see Definition 82 above). For $\alpha < \kappa$, let

$$M_\eta^\alpha = \bigcup_{t \in J_\eta^\alpha} a_t \cup \bigcup_{i < \alpha} b_i$$

and similarly for η' :

$$M_{\eta'}^\alpha = \bigcup_{t \in J_{\eta'}^\alpha} a_t \cup \bigcup_{i < \alpha} b_i.$$

Let C be the cub set of those $\alpha < \kappa$ such that $F \upharpoonright M_\eta^\alpha$ is onto $M_{\eta'}^\alpha$ and for all $i < \alpha$, $B_i \subset M_\eta^\alpha$ and $B'_i \subset M_{\eta'}^\alpha$, where $(\bigcup_{t \in J_{\eta'}^\alpha}, (b'_i, B'_i)_{i < \alpha})$ is in the construction of $M_{\eta'}$. Then we can find $\alpha \in \lim C$ such that in J_η there is t^* satisfying (a)–(c) below, but in $J_{\eta'}$ there is no such t^* :

- (a) $\text{dom}(t^*) = \omega$,

- (b) $t^* \notin J_\eta^\alpha$,
- (c) for all $\beta < \alpha$ there is $n < \omega$ such that $t^* \upharpoonright n \in J_\eta^\alpha \setminus J_\eta^\beta$.

Note that

($\star\star$) if $\alpha \in C$ and $c \in M_\eta^\alpha$, there is a finite $D \subset \bigcup_{t \in J_\eta^\alpha} a_t$ such that $(t(c, \bigcup_{t \in J_\eta} a_t), D) \in F_\omega^f$,

Let $c = F(a_{t^*})$. By the construction we can find finite $D \subset M_{\eta'}^\alpha$, and $X \subset J_{\eta'}$ such that

$$\left(t(c, M_{\eta'}^\alpha \cup \bigcup_{t \in J_{\eta'}} a_t^{\eta'}), D \cup \bigcup_{t \in X} a_t^{\eta'} \right) \in F_\omega^f.$$

But then there is $\beta \in C$, $\beta < \alpha$, such that $D \subset M_{\eta'}^\beta$ and if $u \leq t$ for some $t \in X$, then $u \in J_{\eta'}^\beta$ (since in $J_{\eta'}$ there is no element like t^* is in J_η). But then using ($\star\star$) and (2), it is easy to see that

$$c \downarrow_{M_{\eta'}^\beta} M_{\eta'}^\alpha.$$

On the other hand, using (1), (2), ($\star\star$) and the choice of t^* one can see that $a_{t^*} \not\downarrow_{M_\eta^\beta} M_\eta^\alpha$, a contradiction. \square

Open Problem. If $\kappa = \lambda^+$, λ regular and uncountable, does equality modulo λ -non-stationary ideal, $E_{S_\lambda^\kappa}$, Borel reduce to T for all stable unsuperstable T ?

7. FURTHER RESEARCH

In this chapter we merely list all the questions that also appear in the text:

Open Problem. Is it consistent that Borel^* is a proper subclass of Σ_1^1 , or even equals Δ_1^1 ? Is it consistent that all the inclusions are proper at the same time: $\Delta_1^1 \subsetneq \text{Borel}^* \subsetneq \Sigma_1^1$?

Open Problem. Does the direction left to right of Theorem 24 hold without the assumption $\kappa^{<\kappa} = \kappa$?

Open Problem. Under what conditions on κ does the conclusion of Theorem 36 hold?

Open Problem. Is the Silver Dichotomy for uncountable κ consistent?

Open Problem. Can there be two equivalence relations, E_1 and E_2 on 2^κ , $\kappa > \omega$ such that E_1 and E_2 are Borel and incomparable, i.e. $E_1 \not\leq_B E_2$ and $E_2 \not\leq_B E_1$?

Open Problem. Is it consistent that $S_{\omega_1}^{\omega_2}$ Borel reduces to $S_{\omega}^{\omega_2}$?

Open Problem. We proved that the isomorphism relation of a theory T is Borel if and only if T is classifiable and shallow. Is there a connection between the depth of a shallow theory and the Borel degree of its isomorphism relation? Is one monotone in the other?

Open Problem. Can it be proved in ZFC that if T is stable unsuperstable then \cong_T is not Δ_1^1 ?

Open Problem. If $\kappa = \lambda^+$, λ regular and uncountable, does equality modulo λ -non-stationary ideal, $E_{S_\lambda^\kappa}$, Borel reduce to T for all stable unsuperstable T ?

Open Problem. Let T_{dlo} be the theory of dense linear orderings without end points and T_{gr} the theory of random graphs. Does the isomorphism relation of T_{gr} Borel reduce to T_{dlo} , i.e. $\cong_{T_{\text{gr}}} \leq_B \cong_{T_{\text{dlo}}}$?

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